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with inspirations from concurrency theory

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**CONTRIBUTIONS TO
DIRECTED ALGEBRAIC TOPOLOGY**
– with inspirations from concurrency theory

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Preface

This doctoral thesis is concerned with contributions to the research field *Directed Algebraic Topology*. It is based on seventeen of the author's published research papers ([1] to [17]) in this area from the period 1998 – 2013.

Directed Algebraic Topology is a quite new research discipline. It tries to modify and twist methodology from “classical” Algebraic Topology to a situation where paths, in general, are no-longer reversible. So far, its main motivation comes from the theory of concurrent processes in theoretical computer science. As our main example, we use a geometric/combinatorial abstract model for concurrent computations. We try to find answers to questions concerned with the space of executions for such a model.

While concurrency has served as a prime motivation, the field has also arisen purely mathematical and computational interest. I have to admit that I personally am most passionate about these mathematical aspects; but I am very happy to see that some of them can, in the hands of others, be transformed into actually useful running algorithms.

My thanks go to friends, collaborators, and colleagues around the world. Particular thanks go to my colleague and companion from Aalborg University in this endeavour, Lisbeth Fajstrup. Furthermore, I wish to thank explicitly our long-time friends and collaborators from the LIST laboratory of the CEA at Saclay/Paris in France, first of all Éric Goubault, Emmanuel Haucourt and Samuel Mimram. All the good colleagues in the ACAT network of the ESF have to be mentioned here, as well.

It is my pleasure to thank my colleagues at the Department of Mathematical Sciences at Aalborg University for providing a friendly, supportive and inspiring professional environment.

Finally, I would like to express my gratitude to my family for love and support throughout many years.

Thesis preface

This print of my thesis is identical with the previous version of the manuscript except for minor changes in the layout and for updated references.

Aalborg, May 2014

Martin Raussen

Thesis papers

This thesis is based on the following seventeen publications, in chronological order:

- [1] L. Fajstrup, É. Goubault, and M. Raussen. Detecting Deadlocks in Concurrent Systems. In D. Sangiorgi and R. de Simone, editors, *CONCUR '98; Concurrency Theory*, volume 1466 of *Lecture Notes in Comp. Sci.*, pages 332 – 347, Nice, France, September 1998. Springer-Verlag. 9th Int. Conf., Proceedings.
- [2] M. Raussen. On the classification of dipaths in geometric models for concurrency. *Math. Structures Comput. Sci.*, 10(4):427–457, 2000.
- [3] É. Goubault and M. Raussen. Dihomotopy as a tool in state space analysis. In S. Rajsbaum, editor, *LATIN 2002: Theoretical Informatics*, volume 2286 of *Lecture Notes in Comput. Sci.*, pages 16 – 37, Cancun, Mexico, April 2002. Springer-Verlag.
- [4] M. Raussen. State spaces and dipaths up to dihomotopy. *Homology, Homotopy Appl.*, 5(2):257–280, 2003.
- [5] L. Fajstrup, É. Goubault, E. Haucourt, and M. Raussen. Components of the fundamental category. *Appl. Categ. Structures*, 12(1):81–108, 2004.
- [6] L. Fajstrup, É. Goubault, and M. Raussen. Algebraic Topology and Concurrency. *Theor. Comput. Sci*, 357:241–278, 2006. Revised version of Aalborg University preprint, 1999.
- [7] M. Raussen. Deadlocks and dihomotopy in mutual exclusion models. *Theoret. Comput. Sci.*, 365(3):247 – 257, 2006.
- [8] R. Wisniewski and M. Raussen. Geometric analysis of nondeterminacy in dynamical systems. *Acta Inform.*, 43(7):501 – 519, 2007.
- [9] M. Raussen. Invariants of directed spaces. *Appl. Categ. Structures*, 15:355–386, 2007.
- [10] U. Fahrenberg and M. Raussen. Reparametrizations of continuous paths. *J. Homotopy Relat. Struct.*, 2(2):93–117, 2007.
- [11] M. Raussen. Reparametrizations with given stop data. *J. Homotopy Relat. Struct.*, 4(1):1–5, 2009.
- [12] M. Raussen. Trace spaces in a pre-cubical complex. *Topology Appl.*, 156(9):1718 – 1728, 2009.
- [13] M. Raussen. Simplicial models for trace spaces. *Algebr. Geom. Topol.*, 10:1683 – 1714, 2010.
- [14] L. Fajstrup, É. Goubault, E. Haucourt, S. Mimram, and M. Raussen. Trace Spaces: An Efficient New Technique for State-Space Reduction. In H. Seidl, editor, *Programming Languages and Systems*, volume 7211 of *Lecture Notes in Comput. Sci.*, pages 274–294. ESOP 2012, Springer-Verlag, 2012.

- [15] M. Raussen. Execution spaces for simple higher dimensional automata. *Appl. Algebra Engrg. Comm. Comput.*, 23:59–84, 2012.
- [16] M. Raussen. Simplicial models for trace spaces II: General higher-dimensional automata. *Algebr. Geom. Topol.*, 12(3):1745–1765, 2012.
- [17] M. Raussen and K. Ziemiański. Homology of spaces of directed paths on Euclidean cubical complexes. *J. Homotopy Relat. Struct.*, 9(1):67–84, 2014.

A brief guide to the papers

This is no attempt to describe the content of the papers in any detail; we just try to explain connections between them and express our view on their relative importance:

The paper [1] was the first of our attempts to use geometrical and combinatorial (rather than topological) reasoning in the investigation of a concurrency problem: the detection of *deadlocks* and of *unsafe regions* for a so-called linear PV program.

The papers [3,4] are precursors for [5] that uses a combination of categorical and topological ideas and methods to define and investigate *components* for models of certain concurrent programs.

The paper [9] attempts to describe and organize in rather great generality the many path spaces that a directed space comes equipped with – and also associated algebraic invariants – in a functorial manner. It proposes a candidate for the title *directed homotopy equivalence*.

The papers [10,11] investigate a peculiar but surprisingly rich topic, the algebra underpinning *reparametrizations* of (directed) paths.

The papers [2,7,12] can be considered as precursors to [13]. The first of them treats first of all the (untypical) 2-dimensional case, the second analyses an eye-opener example in 3D, whereas the last investigates general topological properties of path and trace spaces.

I personally view the paper [13] as the most significant contribution. It shows a way to model path and trace spaces *simplicially* (or combinatorially) and gives, for the first time to my knowledge, an algorithmic way to calculate *algebraic topological invariants* of spaces of directed paths given a decent description of the state space. The implementation of the algorithm from [13] and an application/extension to a case with directed loops is the main topic of the paper [14].

The methods from [13] have afterwards been modified and generalized in the papers [15,16] so that they – at least in principle – can be used to identify spaces of executions of general *Higher Dimensional Automata* up to homotopy equivalence by simplicial complexes.

The last paper [17] has its origin in a frustration over the fact that the algorithm designed in [13] resulted in an all too large simplicial complex in a quite simple interesting case (a directed torus with a hole). More delicate homotopy theoretical tools helped to overcome this problem in this particular case; they will hopefully show to be useful in greater generality.

The paper [8] is a bit of an outlier. It builds on Wisniewski's phd-thesis – supervised by me – that investigates almost flow-lines (with respect to a vector field) as d-paths. It uses, first of all, Morse theoretic tools.

The paper [6] is a survey article; a first version had been published as an Aalborg University preprint already in 1999. Though quite out of date and deserving an update, it still gets citations today.

Full list of publications

Mathematical Papers

In *reverse* chronological order, based on Mathematical Reviews and Zentralblatt für Mathematik und ihre Grenzgebiete:

- [1] M. Raussen and K. Ziemiański. Homology of spaces of directed paths on euclidean cubical complexes. *J. Homotopy Relat. Struct.*, 9(1):67–84, 2014.
- [2] M. Raussen. Simplicial models for trace spaces II: General higher-dimensional automata. *Algebr. Geom. Topol.*, 12(3):1745–1765, 2012.
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- [4] L. Fajstrup, É. Goubault, E. Haucourt, S. Mimram, and M. Raussen. Trace Spaces: An Efficient New Technique for State-Space Reduction. In H. Seidl, editor, *Programming Languages and Systems*, volume 7211 of *Lecture Notes in Computer Sci.*, pages 274–294. ESOP 2012, Springer-Verlag, 2012.
- [5] M. Raussen. Simplicial models for trace spaces. *Algebr. Geom. Topol.*, 10:1683–1714, 2010.
- [6] M. Raussen. Trace spaces in a pre-cubical complex. *Topology Appl.*, 156(9):1718–1728, 2009.
- [7] M. Raussen. Reparametrizations with given stop data. *J. Homotopy Relat. Struct.*, 4(1):1–5, 2009.
- [8] U. Fahrenberg and M. Raussen. Reparametrizations of continuous paths. *J. Homotopy Relat. Struct.*, 2(2):93–117, 2007.
- [9] M. Raussen. Invariants of directed spaces. *Appl. Categ. Struct.*, 15(4):355–386, 2007.
- [10] R. Wisniewski and M. Raussen. Geometric analysis of nondeterminacy in dynamical systems. *Acta Inf.*, 43(7):501–519, 2007.
- [11] M. Raussen. Deadlocks and dihomotopy in mutual exclusion models. *Theor. Comput. Sci.*, 365(3):247–257, 2006.
- [12] L. Fajstrup, M. Raussen, and É. Goubault. Algebraic topology and concurrency. *Theor. Comput. Sci.*, 357(1–3):241–278, 2006.
- [13] L. Fajstrup, M. Raussen, É. Goubault, and E. Haucourt. Components of the fundamental category. *Appl. Categ. Struct.*, 12(1):81–108, 2004.
- [14] M. Raussen. A second look at normal curvature. *Normat*, 51(2):59–62, 2003.
- [15] M. Raussen. State spaces and dipaths up to dihomotopy. *Homology Homotopy Appl.*, 5(2):257–280, 2003.
- [16] É. Goubault and M. Raussen. Dihomotopy as a tool in state space analysis. Rajsbaum, Sergio (ed.), *LATIN 2002: Theoretical informatics*. 5th Latin American symposium, Cancun, Mexico, April 3–6, 2002. Proceedings. Berlin: Springer. *Lecture Notes in Comput. Sci.* 2286, 16–37, 2002.

- [17] M. Raussen. Symmetries on manifolds, deformations and rational homotopy: A survey. Bak, Anthony (ed.) et al., Current trends in transformation groups. Dedicated to the memory of Professor Katsuo Kawakubo. Dordrecht: Kluwer Academic Publishers. K-Monogr. Math. 7, 167-179, 2002.
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- [34] M. Raussen and L. Smith. A geometric interpretation of sphere bundle boundaries and generalized J-homomorphisms with an application to a diagram of

- I. M. James. *Q. J. Math., Oxf. II. Ser.*, 30:113–117, 1979.
- [35] M. Raussen. Hurewicz isomorphism and Whitehead theorems in pro-categories. *Arch. Math.*, 30:153–164, 1978.

Interviews and articles in journals of mathematical societies

In reverse chronological order:

- [36] M. Raussen and Chr. Skau. Interview with Abel laureate Pierre Deligne. *Eur. Math. Soc. Newsl.*, 89:15–23, 2013. Reprinted in Notices Amer. Math. Soc. **61** (2014), no. 2, 177–185.
- [37] M. Raussen and Chr. Skau. Interview with Abel laureate Endre Szemerédi. *Eur. Math. Soc. Newsl.*, 85:39–48, 2012. Reprinted in Notices Amer. Math. Soc. **60** (2013), no. 2, 221–231.
- [38] M. Raussen. Computational topology [book review]. *Eur. Math. Soc. Newsl.*, 82:58–59, 2011.
- [39] M. Raussen and Chr. Skau. Interview with Abel laureate John Milnor. *Eur. Math. Soc. Newsl.*, 81:31–40, 2011. Reprinted in Notices Amer. Math. Soc. **59** (2012), no. 3, 400 – 408.
- [40] M. Raussen and Chr. Skau. Interview with Abel laureate John Tate. *Eur. Math. Soc. Newsl.*, 77:41–48, 2010. Reprinted in Notices Amer. Math. Soc. **58** (2011), no. 3, 444 – 452.
- [41] M. Raussen and Chr. Skau. Interview with Abel laureate Mikhail Gromov. *Eur. Math. Soc. Newsl.*, 73:19–30, 2009. Reprinted in Notices Amer. Math. Soc. **57** (2010), no. 3, 391 – 403.
- [42] M. Raussen. Interview with Jacques Dixmier. *Eur. Math. Soc. Newsl.*, 72:34–41, 2009.
- [43] M. Raussen. Invitation to topological robotics [book review]. *Eur. Math. Soc. Newsl.*, 72:46–47, 2009.
- [44] M. Raussen and Chr. Skau. Interview with John G. Thompson and Jacques Tits. *Eur. Math. Soc. Newsl.*, 69:31–38, 2008. Reprinted in Notices Amer. Math. Soc. **56** (2009), no. 4, 471 – 478.
- [45] M. Raussen and A. Valette. An interview with Beno Eckmann. *Eur. Math. Soc. Newsl.*, 66:31–37, 2007. Reprinted in math.ch/100 (Eds: B. Colbois, C. Riedtmann, V. Schroeder), EMS publishing house (2010), 389–401.
- [46] M. Raussen and Chr. Skau. Interview with Srinivasa Varadhan. *Eur. Math. Soc. Newsl.*, 65:33–40, 2007. Reprinted in Notices Amer. Math. Soc. **55** (2008), no. 2, 238–246.
- [47] U. Persson and M. Raussen. Two mathematicians forced to resign at Uppsala University, Sweden. *Eur. Math. Soc. Newsl.*, 64:15, 2007.
- [48] M. Raussen and Chr. Skau. Interview with Abel Prize recipient Lennart Carleson. *Eur. Math. Soc. Newsl.*, 61:31–36, 2006. Reprinted in Notices Amer. Math. Soc. **54** (2007), no. 2, 223–229 and Mitt. Dtsch. Math.-Ver. **14** (2006), no.4, 206–212.
- [49] M. Raussen and Chr. Skau. Interview with Peter D. Lax. *Eur. Math. Soc. Newsl.*, 57:24–31, 2005. Reprinted in Notices Amer. Math. Soc. **53** (2006), no. 2, 223–229.
- [50] M. Raussen and Chr. Skau. Interview with Michael Atiyah and Isadore Singer. *Eur. Math. Soc. Newsl.*, 53:24–30, 2004. Reprinted in Notices Amer. Math. Soc. **52** (2005), no. 2, 225–233 and Mitt. Dtsch. Math. Ver. **12** (2004), no. 4, 272–281.
- [51] M. Raussen and Chr. Skau. Interview with Jean-Pierre Serre. *Eur. Math. Soc. Newsl.*, 49:18–20, 2003. Reprinted in Notices Amer. Math. Soc. **51** (2004), no. 2, 210–214 and Nieuw Arch. Wisk (5) **5** (2004) no. 1, 38–41.

Moreover, interviews with Danish mathematicians Ebbe Thue Poulsen, Bent Fuglede, Tobias Colding and Ib Madsen and a few articles were published in the journal *Matilde* of the Danish Mathematical Society (in Danish).

Interviews with the Abel prize recipients were broadcasted by the 2nd chain of Norwegian TV (kunnskabpskanalen) and are archived on the Abel Prize website.

CHAPTER 1

Introduction

1.1. Motivations. Background

1.1.1. A personal report. During my education as a mathematician, I was primarily trained within differential and algebraic topology. This is clearly visible from the older entries in my list of publications: Until the middle of the 1990s, my research was focused on various aspects of algebraic topology, often on problems concerning group actions on manifolds, some of them quite technical at the end of the day.

Being based at Aalborg University with an emphasis on engineering and applied sciences, I felt after all quite alone – even after another topologist, Dr. Lisbeth Fajstrup, had been appointed in 1992. This feeling and a combination of encouragement and pressure from leading people at the department led to a look for alternative research directions, not too far from our experiences. Such an alternative became apparent when the two of us participated in a weeklong workshop at the Isaac Newton Institute in Cambridge in late 1995 under the title *New Connections between Mathematics and Computer Science*, cf Gunawardena [Gun96].

This workshop was held in a very nice, open and fruitful atmosphere with a variety of stimulating talks, including very famous speakers like M. Gromov and S. Smale. But there were other talks that turned out to be more decisive for our work. Apart from lectures by John Baez (*n-categories in logic, topology and physics*), Yves Lafont (*Homological methods and word problems*), I would like to emphasize in particular

- Éric Goubault, *Scheduling problems and homotopy theory*
- Sergio Rajsbaum, *On the decidability of a distributed decision task*

that introduced us for the first time to the possibility of applying topological methods for purposes in concurrency theory and in distributed systems theory.

1.1.2. Concurrency and distributed computing. To say it very briefly¹, *concurrency* in computer science means a property of systems in which several computations are executing simultaneously, and potentially interacting with each other. Concurrent systems open up for faster algorithms, but the number of possible execution paths (schedules) in such a system can be extremely large and the resulting outcome may be indeterminate. Methods are sought to identify schedules that do produce results (not ending in a deadlock) and that lead (by construction) to correct – or at least tolerable – results.

Distributed systems, studied in *distributed computing* consist of multiple autonomous computers communicating through a computer network in order to

¹A comprehensive survey over a wealth of models in concurrency is given in a chapter of the Handbook of Logic in Computer Science by Winskel and Nielsen [WN95].

achieve a common goal. Typically, a problem is divided into many tasks, each of which is solved by one or more computers, communicating with each other by a variety of protocols. What sorts of problems can be solved in distributed architectures – possibly assuming that a number of participating computers may fail to work, without notice to the others? An intriguing introduction to topological methods had at the time just appeared (cf Herlihy and Rajsbaum [HR95], in particular; moreover Herlihy and Shavit [HS99]).

The two talks mentioned above advocated that these two disciplines, certainly related to each other, but with slightly different goals to achieve, may benefit from a perspective from combinatorial/algebraic topology and showed indications and some results in that direction. These prospects made an impression on us. We began to think, in particular, about how to detect *deadlocks* in semaphore models, cf Chapter 2 of this thesis. We made soon personal connections with Éric Goubault; at the time employed at the ENS in Paris, now a professor at CEA Saclay and at Ecole Polytechnique. This encounter started a very fruitful collaboration that has been ongoing ever since.

It turned out that we would try to take inspiration from methods in combinatorial and algebraic topology that we knew; but it was not possible to apply those directly, before “twisting” them. That twist consisted in taking *directedness* properties serious. No longer are all continuous paths allowed, only *directed* paths, reflecting that the time flow in the execution of a schedule is not reversible. This fact makes it more difficult to exhibit suitable algebraic topological invariants describing phenomena of interest. At least, one has to get involved with *categories* instead of groups. These were the first indications for a need for a systematic investigation of methods for and properties of *Directed Algebraic Topology*. It should be mentioned in passing that other topological methods (order topologies etc) had been applied previously in Computer Science, in particular in *domain theory*.

1.2. Collaboration

While algebraic topology for a long time had the reputation of an exclusive and very pure mathematical discipline, more and more areas of application have popped up during the last fifteen to twenty years, cf also Chapter 8.

1.2.1. Workshops, conferences, networks. One of the first initiatives to collect researchers with an interest in applying methods from algebraic topology to problems in Computer Science after the conference at the Newton Institute in 1995, was our own series of modest workshops called GETCO (Geometric and Topological Methods in Computer Science): the first of those was held in 1999 at Aalborg University and followed up by a series of similar workshops lasting between a day and a week; several times attached to conferences of the CONCUR or DISC communities in Computer Science.

At a much larger scale, I would like to mention the very inspiring conference series ATMCS (Algebraic Topological Methods in Computer Science), that has been organized five times, lastly in 2012 under the title Applied and computational topology. Moreover workshop series at Schloss Dagstuhl, Germany and dedicated conferences at MSRI, Berkeley, USA, Oberwolfach, Germany, the Fields Institute, Toronto, Canada, and BIRS, Banff, Canada.

On the European arena, collaboration on applied aspects of algebraic topology gained force in the recent Research Networking Programme ACAT – Applied and Computational Algebraic Topology – in the framework of the European Science Foundation. This programme that lasts from 2011 to 2015, has some funds to support conferences, workshops and summer schools within the field; moreover it gives grants to visits between collaborating partners. I am the chair of the steering committee of that network.

1.2.2. Acknowledgements. It would have been impossible to achieve substantial progress without a network of people who have been interested in the research line taken and with whom I have had the pleasure to collaborate for a while or also on an almost permanent basis. It is impossible to mention them all, but I need to give special thanks to

- Lisbeth Fajstrup, colleague at the Department of Mathematical Sciences at Aalborg University, a long term partner and a coauthor in this endeavour; giving inspiration and – very important – ready to listen whatever I had on the agenda
- Éric Goubault, CEA/Saclay and Ecole Polytechnique, France, who would explain many times to us many of the Computer Science aspects of our work; moreover an important source of inspiration and coauthor
- his colleagues Emmanuel Haucourt and Samuel Mimram who joined a little later
- my Ph.D.-students Ulrich Fahrenberg, Rafael Wisniewski and John-Josef Leth for inspiring discussion during an extended period
- Maurice Herlihy who inspired us many times with his well-planned and funny talks on decision problems in distributed computing and who was my host during a three weeks visit at the Computer Science Department at Brown University, Providence, RI, USA, in 2000
- Marco Grandis (Genova University) for deep interest in the subject and for writing the first book Grandis [Gra09] on Directed Algebraic Topology
- Krzysztof Ziemiański (Warsaw University) for recent collaboration.

Thanks are also due to the referees of my work in the area who have often given me indications how to make my drafts more consistent and/or more readable.

1.3. Not more than a survey!

This thesis can only give a quick guide through the material described in much more detail in the articles that are submitted together with it (cf the list of thesis papers right in the beginning). It is written in retrospect; only some highlights are dealt with and deviations from the main route are deliberately kept obscure.

Most of the work is not technically deep or sophisticated. Beginning with a new research interest meant that many concepts had to be developed if not from scratch, then from only a few basic definitions. The only exception is the development of combinatorial models for trace spaces, cf Chapter 7, that nevertheless, from a mathematical perspective, stands on the shoulders of well-developed techniques.

For proofs of results, the reader is referred to the original papers. Only a few particularly important proofs that have a particular impact on this story have been detailed in this thesis. Emphasis is put on more recent work – that of course uses

insights from previous articles. The selection of topics has mainly been made from a *mathematical* perspective.

Topology and order for semaphore models

2.1. Semaphore models

2.1.1. History. First notions and examples. The following notes on the (pre)-history of the subject are mainly drawn from Goubault [Gou00]; they describe the initial motivation for our and for related work.

An option for scheduling the access of several processes to shared resources is by *semaphores*: Each resource is provided with a semaphore. A schedule has to obey to the rule that, at any given time, only one (in the case of mutual exclusion) or at most a fixed number k of processes – called the *arity* of that resource – can acquire a lock to a given resource. It has to relinquish the lock having finished working with the resource. It is well-known (even in daily life) that “bad” schedules can lead to deadlock states from which the combined execution has no way to proceed.

Semaphore methods can be given a description that has an inherently geometric flavour: The so-called “progress graph” was first introduced in the literature in Shoshani and Coffman [SC70] and Coffman et al. [CES71]. The famous Dutch computer scientist Edsger W. Dijkstra [Dij68] had given an abstract semantics for handling mutual exclusion: Each (deterministic sequential) process Q_i gives rise to a sequence $R^1 a_i^1 \dots R^{n_i} a_i^{n_i}$ with $R^j = P, V$ and a_j one of the shared objects; P (*prolaag* in Dutch; procure?) means acquiring a lock, V (*verhogen* in Dutch; vacate?)¹ means relinquishing it again.

Simple test examples can be found in the early literature [SC70] and [CES71]; they are attributed there to Dijkstra. The following two examples must suffice here: The first (the “*Swiss flag*”) consists of two processes $T_1 = Pa.Pb.Vb.Va$ and $T_2 = Pb.Pa.Va.Vb$ competing for resources a and b and gives rise to the two dimensional progress graph of Figure 1.

The second (of which the first is a special case) is known under the name *Dining Philosophers*: Here n philosophers T_1, \dots, T_n at a round dining table compete for n resources (forks) a_1, \dots, a_n according to the schedules $T_i = Pa_i.Pa_{i+1}.Va_i.Va_{i+1}$ (with $a_{n+1} = a_1$) giving rise to an n -dimensional progress graph. It is obvious that trouble (a deadlock) arises when all philosophers start to pick up their left forks at the same time.

For a general linear progress graph, each individual process executes linearly on a (time) unit interval $I = [0, 1]$ on which the P and V actions are marked as an ordered sequence. If there are n processes involved, the state space X consists of a hypercube I^n (each point has n coordinates, one for every process) from which a *forbidden region* F has been deleted: $X = I^n \setminus F$. The forbidden region consists of points for which more than one (generally more than the arity k) coordinates

¹These terminology explanations come from the Wikipedia page on semaphores.

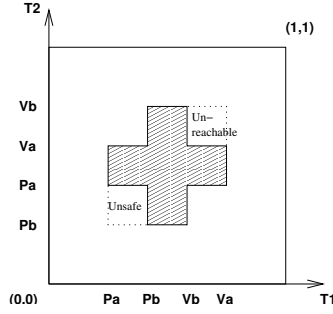


FIGURE 2.1. The Swiss flag as example of a 2D progress graph

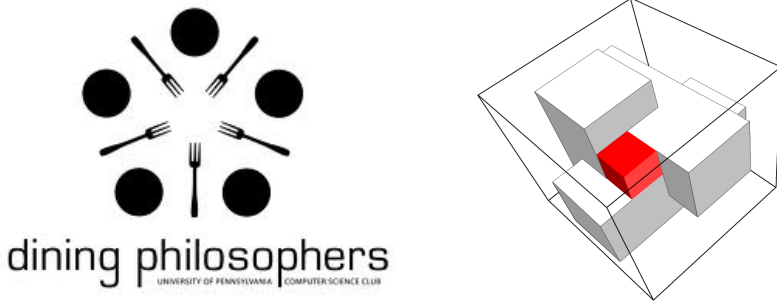


FIGURE 2.2. Left: Dining philosophers. Right: Forbidden and unsafe regions for the three dining philosophers protocol

are situated inbetween a Pa_i and a Va_i with the *same* a_i . It is easy to see that the forbidden region in the semaphore case is a union of (open) higher dimensional *isothetic* rectangles – with facets parallel to the coordinate hyperplanes.

A joint schedule in a progress graph corresponds to a path p in \mathbf{R}^n joining the compound start state, the lowest vertex $\mathbf{0} \in I^n$ to the compound end state, the upmost vertex $\mathbf{1} \in I^n$. The projections of every such path to one of the axes (describing the execution of the corresponding process with the locking instructions on its way) has to be *non-decreasing* since an execution does not run backwards in time. Moreover, such a path may not enter the forbidden region $F \subset I^n$.

The characteristic features of such a *dipath* [FGR06] – di for directed – $p : I \rightarrow I^n$, are hence that

- (1) $p(t)$ avoids the forbidden region F for all t ;
- (2) all projections $p_i : I \rightarrow I$ are non-decreasing.

We will often also fix start and end points.

Deadlocks (originally called “deadly embrace” by Dijkstra) occur in many schedules; these are states (points) from which no *directed* path can proceed without immediately entering the forbidden region F . Furthermore, there may occur “unsafe regions” (a dipath entering the unsafe region cannot reach a final state without entering the forbidden region, cf. Figure 2.1 and Figure 2.2) and “unreachable regions” (that no dipath starting from the compound start state can ever enter). Deadlocks, unsafe regions and unreachable regions had, to a certain extent,

already been analysed and described in this context (including interesting test examples) in many articles, see eg Lipski and Papadimitiou [LP81] or Carson and Reynolds [CR87].

If the processes (on the axes of the progress graph) run without or with restricted coordination, many possible schedules - and in a progress graph even infinitely many - will arise as dipaths. The following is an important insight indicating that topology might have a role to play: Two schedules will yield the same result (of a joint calculation or whatever) if the two respective dipaths can be connected to each other by a *one-parameter deformation* of dipaths (avoiding the forbidden region along that deformation); the reason for this will be explained in detail in Section 3.2. In topology, one-parameter deformations are called homotopies; we call one-parameter deformations that respect directedness *dihomotopies*. Loosely speaking, dihomotopic dipaths acquire their locks and relinquish them in the same order, at least for semaphores of arity one. In the paper Fajstrup et al. [FGR06], an example of a simple *PV*-program is given (with simple calculation steps between accesses to pieces of shared memory) showing that results in general will be different for dipaths that are *not* dihomotopic. The schedules in this example are in fact two dipaths circumvening the forbidden region in Figure 1 in two different senses.

2.1.2. Discrete versus continuous. This gives room for speculation. At first sight, it may seem strange to replace a large discrete state space (as is common in concurrency theory, e.g., a graph, that is contained in the product of the directed graphs describing the actions for each of the processes) by an *infinite* state space. That directed graph can be considered as the 1-skeleton of a subdivided progress graph (cf. Section 3.1.4). Higher-dimensional rectangles in the progress graph describe additional information, i.e., higher coordination (independence relations) between actions of the individual processes.

For realistic examples, the state space is built either as a discrete product of general *directed graphs* (with branchings, mergings and loops; not only linear graphs) or as the topological product of their *geometric realizations*, in both cases after deleting a forbidden region. This situation is more complicated than the linear one and will be dealt with later.

Anyway, when the number of states and/or the number of processes increases, a discrete product will suffer from (combinatorial) state space explosion. Moreover it is quite difficult to determine which of the directed paths (even much bigger in number) in such a product graph are equivalent to each other and which not.

Deformations (homotopies) are well-studied in algebraic topology, and we attempt to use methods from this area to reason and to do calculations concerning the space of directed paths (up to homotopy) in the model. The “detour” through the continuous models seemingly allows a quicker (or more comprehensible) path to a determination of equivalence classes of execution paths.

2.1.3. Example of an application: data base theory. A nice example where topological reasoning gives rise to insights was originally described by J. Gunawardena in [Gun94] and later made more rigorous in Fajstrup et al. [FGR06]: In data base engineering, one uses often *2-phase locked protocols* with the following defining property: Each *PV*-protocol for each of the processes needs to acquire *all* locks before these are *all* relinquished (possibly in a different order): $P \dots P.V \dots V$.

It can then be shown that every schedule using this strategy is *serializable*, i.e., equivalent to a serial schedule characterized by the property: One process at a time! The schedule is a concatenation of the schedules of the individual processes (in some order); no interleaving takes place.

This is important since the results of serial schedules can be checked and understood quite easily. The geometric/topological picture that corresponds to a 2-phase locked protocol yields a progress graph in which the forbidden region has a *center region* from which one may deform all dipaths inductively to dipaths on the *1-skeleton* of the hypercube I^n . One has to be careful to make sure that the deformation is through *dipaths* at any time – this is the contribution of Fajstrup et al. [FGR06]. But still the argument given (and certainly the intuition behind it) seems to be far easier to comprehend than arguments of a merely discrete type.

2.2. Detection of deadlocks and of unsafe regions

First joint work describing how to find deadlocks and unsafe directions in a progress graph can be found in the preprint Fajstrup and Raussen [FR96]. This work was completed in collaboration with Éric Goubault (Fajstrup et al. [FGR98b, FGR98a]) with the description of a running implementation of the detection algorithm.

2.2.1. A combination of combinatorial and geometric insight leads to an algorithm. How can one find *deadlock states* in the n -dimensional state space $X = I^n \setminus F$ with the combinatorial input given by the hyperrectangles R^i making up the forbidden region F ? Those hyperrectangles are products of intervals $R^i = \prod_{j=1}^m [a_j^i, b_j^i] \subset I^n$ (between a P -action at a_j^i of the i th process and the corresponding V -action at b_j^i). If n such hyperrectangles intersect (generically), then the lowest corner of the intersection hyperrectangle is a deadlock state! Whichever process progresses, it will have to enter one of the hyperrectangles. This corner point is coordinatewise given by the *maximal* coordinates of the lower corners a_j^i of the contributing hyperrectangles.

A simple illustrating example consists of the walls and the ceiling of a rectangular room making up parts of the forbidden region: No way from the lower corner of the intersection! The corners obtained in this way are the only deadlock points in the interior of I^n . Deadlocks on the boundary of I^n can be found by the same mechanism after having extended the given hyperrectangles touching the boundary and added hyperrectangles representing the boundary; for details see Fajstrup et al. [FGR98b, FGR98a].

It is even more important to describe the *unsafe regions* that schedules had better avoid. It turned out, that these can be determined step by step using similar ideas. For the first step, note that *below* a deadlock state, there is a hyperrectangle whose lower corner has as coordinates the *second largest* among all the lower corner coordinates a_j^i of the participating hyperrectangles. It is easy to convince yourself that dipaths in this hyperrectangle cannot escape without entering the forbidden region.

In subsequent steps, one adds the unsafe hyperrectangles found so far to the forbidden region; this may then give rise to *additional* unsafe hyperrectangles. It is not difficult to see (and it is shown in Fajstrup et al. [FGR98a]) that the algorithm

thus described ends after finitely many steps with a complete description of the unsafe regions.

Although the algorithm can be described entirely in discrete terms without any use of topological machinery, it is certain that we would not have found it (and it would be hard to explain it) without geometric thinking and intuition.

2.2.2. Implementation issues. The algorithm for detection of deadlocks and unsafe regions was implemented by Éric Goubault as a C-program. It had essentially to manipulate intersections and unions of isothetic n -rectangles from a given list and to implement the effect of adding an additional n -rectangle. In a first step, the initial list of forbidden n -rectangles is established, the second step works out an array of intersection rectangles (including as a special case those with deadlocks at corner points), and the third adds pieces of the unsafe regions, recursively.

The total complexity of the algorithm reflects the geometric complexity of the forbidden region, i.e., the number of intersections of n -rectangles in the forbidden region. The latter express the degree of synchronization of the processes. In most test cases, the implementation gave very competitive results compared to other approaches; in particular, in high dimensions, with many participating processes; some details are given in Fajstrup et al. [FGR98a].

These first tools have since been extended into a multi-purpose abstract interpretation based static analyzer ALCOOL by our partners at CEA/LIST, cf eg Goubault and Haucourt [GH05] and Fajstrup et al [FGH⁺12].

2.3. Outlook and discussion

As mentioned earlier, the state space X of a concurrent program will, in general, be modeled as (topological) product of directed graphs from which a subcomplex corresponding to the forbidden region F has to be deleted: $X = (\Gamma_1 \times \cdots \times \Gamma_n) \setminus F$. The individual graphs and thus also the state space X may contain *directed loops*. Still the forbidden region can be understood as a union of generalized hyperrectangles (with identifications on the boundary, giving rise to cylinders or tori). The detection of deadlock points as corners of intersections of n -rectangles is essentially unchanged, but the detection of unsafe points is more tricky: States that seem to be unsafe might be able to escape after several “rounds”. The effects of this delooping have been described and investigated in Fajstrup [Faj00] and Fajstrup and Sokołowski [FS00]. They were the original motivation for extending the theory of coverings from algebraic topology to so-called dicoverings, cf Fajstrup [Faj03] which turned out to be much more sophisticated.

In another direction, it turned out that the description of deadlocks and unsafe (and, in an analogous manner, also of unreachable) regions is a helpful step in the classification of dipaths in progress graphs up to dihomotopy in later work (cf Raussen [Rau00, Rau06, Rau10]); this will be taken up in subsequent sections.

Directed spaces and directed homotopy

3.1. Directed spaces

3.1.1. Partially ordered spaces. Apparently the first attempt to combine order and topology in a systematic way can be found in Nachbin's monography [Nac65]. The following definition *connecting topology and order* is particularly important:

DEFINITION 3.1.1. Let X denote a topological space and let $\leq \subset X \times X$ denote a partial order (reflexive, transitive and anti-symmetric) on X . The pair (X, \leq) is called a *partially ordered space* (or *po-space* for short) if \leq is a *closed* subset of $X \times X$.

In particular, if x_n, y_n denote sequences in X with $x_n \leq y_n, n \in \mathbf{N}$, with limits $x = \lim x_n, y = \lim y_n$, then $x \leq y$. For example, the standard (coordinatewise) partial order \leq on \mathbf{R}^n makes (\mathbf{R}^n, \leq) a po-space. A closed subspace (like the Swiss flag example from Figure 1) inherits a po-structure.

Another stimulating monography involving order notions is Penrose's [Pen72] that deals with questions in relativity theory from the viewpoint of differential geometry. A space-time is seen as a $4=(3+1)$ -dimensional manifold with a Lorentzian metric of index 1. In particular, the tangent bundle contains a "bundle of cones" consisting of *causal* resp. *time-like* tangent vectors. One studies then properties of *causal* and of *time-like* curves (with causal, resp. time-like tangent vectors all along) in combination with the differential geometry of the underlying manifold, eg with the aim to investigate properties of black holes. Several notions from Penrose's book [Pen72] have been useful in connection with our work although they often had to be modified in order to fit for our purposes.

In both cases mentioned above, one considers most often not *all* curves in the topological space; only curves that play together well with order properties are relevant. In a *po-space*, a directed path $p : I = [0, 1] \rightarrow X$ has to preserve orders, i.e., has to satisfy:

$$(3.1) \quad t_1 \leq t_2 \Rightarrow p(t_1) \leq p(t_2).$$

As soon as one considers spaces that contain interesting (from the point of view of order) *loops*, po-spaces are too rigid: Property (3.1) cannot be satisfied for any partial order \leq along a non-constant loop p (with $p(0) = p(1)$) since one obtains for any $x = p(t) \neq p(0)$: $p(0) \leq x \leq p(1) = p(0)$ contradicting anti-symmetry. On the other hand, state spaces with loops occur naturally in applications, since most relevant programs contain loops.

There are several reasonable approaches to widen the definition of a po-space:

3.1.2. Lpo-spaces. The one originally taken by us was an approach using charts (compare the definition of manifolds) as in the following definition:

DEFINITION 3.1.2. (Fajstrup et al. [FGR06])

- (1) A covering \mathcal{U} of a topological space X by open sets U with partial orders \leq_U is an atlas of an *lpo-space* (locally partially ordered) if, for every $x \in X$, there is a non-empty neighbourhood $W(x)$ such that

$$y \leq_U z \Leftrightarrow y \leq_V z \text{ for all } U, V \in \mathcal{U}, y, z \in U \cap V \cap W(x).$$

- (2) Two atlas \mathcal{U} and \mathcal{V} define equivalent lpo-structures if their union $\mathcal{U} \cup \mathcal{V}$ defines an lpo-structure in X .

For example, (counter-clockwise) rotations cannot be used to give the circle $X = S^1$ a po-structure, but they give rise to a perfectly well-defined lpo-structure. Moreover, one may ask a *dimap* to preserve partial local orders (cf Fajstrup et al. [FGR06]) and in particular investigate *dipaths* and *diloops* in lpo-spaces as dimaps with domain the ordered interval \tilde{I} and the lpo-space \tilde{S}^1 .

Directed paths are the essential object of our study – they correspond to the execution paths in the model; the (partial) order on the state space is secondary. This makes the study of lpo-spaces by themselves a doubtful goal. In fact, for a space with a partial order, one may define a (coarser) partial order by:

$$x \preceq y \Leftrightarrow \exists \text{ a dipath } p : I \rightarrow X \text{ such that } x = p(0), y = p(1).$$

Then, x and y are no longer related unless they are so via a directed path. For example, the region under the Swiss flag (Figure 1) becomes then completely unrelated to the region above it. On the other hand, it is not always clear that the new partial order \preceq thus obtained is closed.

A more serious drawback of lpo-spaces arises from categorical considerations. An investigation of (finite) limits and colimits shows that these need not always exist; if they exist they need not be the limits or colimits that arise in the category of topological spaces (under the forgetful functor). This is maybe not that surprising; it parallels the bad behaviour of the category of manifolds under limits and colimits. There have been attempts to reconcile lpo-spaces with model categories (cf Bubenik and Worytkiewicz [BW06]); but those seemingly remained without further practical applications.

3.1.3. Streams. These categorical problems have been overcome by introducing the more flexible *streams* in the work of Krishnan [Kri09]. Roughly speaking, a stream is a topological space with consistent *preorders* on the open sets, a so-called *circulation*. Consistency means that the preorder on an open set is equal to the transitive closure of the preorders on its open subsets. It is shown in Krishnan [Kri09] that streams and stream maps (preserving the local preorders) form a complete and cocomplete category.

Lpo-spaces are more special than streams: Local anti-symmetry is *not* required for streams. In particular, streams may have vortices, ie they may allow for arbitrarily small directed loops.

3.1.4. Cubical complexes. Higher-Dimensional Automata. Higher-Dimensional Automata (HDA) were originally introduced and studied by Pratt [Pra91] and van Glabbeek [vG91] as combinatorial models extending the progress graphs from Section 2.1.1. Roughly speaking, the concurrent parallel execution of one step taken by each of n individual processes is modelled by an n -box or n -cube \square_n if it is independent of orderings among the processes (and perhaps also subdivisions).

An n -cube can thus be seen as the state space for all possible interleavings of the $n!$ directed paths on its one-skeleton. The presence of such a cube in a complex indicates that the order of the partial executions on the 1-skeleton (or even of partial interleavings) is insignificant. On the other hand, if nothing is known about such independence relations, only the 1-skeleton of such a box appears in the HDA.

Knowledge about *partial independence* can be encoded by considering a *sub-complex* of the n -box containing its 1-skeleton. A subcube $\square_k \subset \square_n$ is included if k -processes execute independently as long as the others have come to a halt. The partial ordering on an k -box (equivalent to \tilde{I}^k) yields a natural directed path structure.

Several such (sub-)boxes (perhaps of varying dimensions; the number of participating processes and the nature of independence relations may vary) can be glued together to yield a Higher-Dimensional Automaton (HDA). Usually, at least the 1-skeleton of such an HDA is equipped with labels, as a generaliation of transition systems in classical concurrency theory. In the following, we shall abstract away from the use of such labels; this is in a sense justified by results of Srba [Srb01] showing – for transition systems – that a labelled transition system can be replaced by an equivalent unlabelled one without loosing expressivity. Let us also mention that van Glabbeek [vG06] later showed that Higher Dimensional Automata compete favourably with other widely used models for concurrent computing, eg Petri nets.

The definition of an HDA and discussions about their properties came closer to well-known mathematics when Éric Goubault found out that the underlying combinatorics and topology is that of a pre-cubical set (also called \square -set, cf. Fajstrup [Faj05], in analogy with the term Δ -set from Rourke and Sanderson [RS71] for a simplicial set without degeneracies).

Pre-cubical sets had been previously investigated in detail by Brown and Higgins in [BH81a, BH81b]. Cubical complexes are already present underlying cubical singular homology in Serre's thesis [Ser51] from 1951. For modern homotopy theory references involving cubical sets see Jardine [Jar02] and Grandis and Mauri [GM03].

In the following section, we shall use \square_n as an abbreviation for the n -cube $I^n = [0, 1]^n$ with the product topology.

DEFINITION 3.1.3. (1) A \square -set or pre-cubical set M is a family of disjoint sets $\{M_n | n \geq 0\}$ with face maps

$$\partial_i^k : M_n \rightarrow M_{n-1}, \quad n > 0, \quad 1 \leq i \leq n, \quad k = 0, 1,$$

satisfying the pre-cubical relations $\partial_i^k \partial_j^l = \partial_{j-1}^l \partial_i^k$ for $i < j$.

(2) A pre-cubical set M is called *non-self-linked* (cf Fajstrup et al. [FGR06]) if, for all $n, x \in M_n$ and $0 < i \leq n$, the $2^i \binom{n}{i}$ iterated faces

$$\partial_{l_1}^{k_1} \cdots \partial_{l_i}^{k_i} x \in M_{n-i}, \quad k_i = 0, 1, \quad 1 \leq l_1 < \cdots < l_i \leq n, \quad \text{are all different.}$$

(3) The geometric realization $|M|$ of a pre-cubical set M is given as the quotient space $|M| = (\coprod_n M_n \times \square_n)_{/\equiv}$ (a cubical complex) under the equivalence relation induced from

$$(\partial_i^k(x), t) \equiv (x, \delta_i^k(t)), \quad x \in M_{n+1}, \quad t = (t_1, \dots, t_n) \in \square_n, \quad 1 \leq i \leq n, \quad k = 0, 1$$

with $\delta_i^k(t) = (t_1, \dots, t_{i-1}, k, t_{i+1}, \dots, t_n)$.

In a *non-self-linked* pre-cubical set, the map $\square_n \simeq \square_n \times e \rightarrow |M|$ is *injective* for every n -cell $e \in M_n$. In particular, every element $m \in |M|$ in the image of this map has *uniquely* determined coordinates in \square_n , cf Fajstrup et al [FGR06]. Moreover, every element $x \in |M|$ has a unique *carrier* cell $e(x) \in M_n, n \geq 0$, such that x comes from an element in the *interior* \square_n^o under the restriction of the quotient map to $\square_n \times e(x)$.

We will make use of particular open sets in $|M|$, the *open stars* of *vertices* in M_0 . The open star $St(x, M)$ of $x \in M_0$ consists of the union of the interiors of all cells of which x is a vertex. It was shown in Fajstrup et al [FGR06], that every such open star inherits a *consistent* partial order from the partial orders on individual cells given by their identification with $\square_n \subset \mathbf{R}^n$ and hence that

PROPOSITION 3.1.4. (Fajstrup et al. [FGR06, Theorem 6.23]) *The po-structure on the cells of a \square -set extends to an lpo-structure on its geometric realization if M is non-selflinked.*

In particular, we know which paths in $|M|$ are to be considered as *directed*; for a down-to-earth description cf Raussen [Rau09b]. The geometric realizations of these pre-cubical sets are arguably the most important class of examples of lpo-spaces for applications. They are not as special as they might look at a first glance: Fajstrup [Faj06] showed that every triangulable space can be realized as a cubical complex. If this cubical complex is free of immersed cubical Möbius bands, then there are consistent choices of directions; if this is not the case, one subdivision suffices to establish a compatible local partial order.

3.1.5. D-paths, d-spaces, d-TOP. A very general approach to directed spaces that plays together well with established techniques in homotopy theory was suggested by Marco Grandis. He launched in Grandis [Gra03a] the idea to take the directed paths (d-paths) as *defining* element of the structure of d-spaces:

As customary, the *concatenation* of two paths $p, q : I \rightarrow X$ in a topological space X is defined by $(p * q)(t) = \begin{cases} p(t) & t \leq 0.5 \\ q(2t - 1) & t \geq 0.5 \end{cases}$.

DEFINITION 3.1.5. (Grandis [Gra03a]) Let X denote a topological space and let $\vec{P}(X) \subset X^I := \{p : I \rightarrow X \mid p \text{ continuous}\}$ – the subset of *d-paths*. The pair $(X, \vec{P}(X))$ is called a *d-space* if

- $\vec{P}(X)$ contains every *constant* path $p_x(t) = x, t \in I; x \in I$;
- The concatenation of two d-paths is a d-path:

$$p, q \in \vec{P}(X) \Rightarrow p * q \in \vec{P}(X);$$

- $p \in \vec{P}(X), \alpha \in I^I$ a *non-decreasing* reparametrization $\Rightarrow p \circ \alpha \in \vec{P}(X)$.

Remark that only *non-decreasing* reparametrizations are part of the structure. Thus, in general the reverse \bar{p} of a d-path p given by $\bar{p}(t) = p(1 - t)$ is not a d-path. On the other hand, a sub-d-path of a d-path is d again. Note two extreme cases:

- $\vec{P}(X)$ consists solely of constant paths.
- $\vec{P}(X) = X^I$ consists of *all* paths.

While the di-paths in an lpo-space X provide it with a d-space structure, the last example above shows that not every d-space arises from an lpo-space.

In particular, antisymmetry in an lpo-space forbids the existence of small (non-constant) loops; but vortices can perfectly arise in a d-space. For example, the d-paths in the plane might consist of paths rotating counterclockwise around the origin. For homotopy theory purposes, it is an advantage that one can give not only the *cylinder* but also the *cone* of a d-space the structure of a d-space; cf. the work of Grandis in [Gra03a, Gra02, Gra09].

A continuous map $f : X \rightarrow Y$ is called a *d-map* if it preserves d-paths, i.e., if $f(\vec{P}(X)) \subseteq \vec{P}(Y)$. We can consider the subcategory $\mathbf{d-TOP} \subset \mathbf{TOP}$ (with d-maps as the morphisms) of the category of topological spaces. It is not difficult to see that the category $\mathbf{d-TOP}$ thus arising has all limits and colimits, cf Grandis [Gra09]; in particular, there is an obvious notion of product of d-spaces.

A correspondence between the category of streams and that of d-spaces has been described and investigated by Ziemiański [Zie12a]. They are related by adjoint functors leading to isomorphisms of categories of “good” streams and of “good” d-spaces. These are very close to the saturated directed spaces of Hirschowitz et al [HHH13].

3.2. Dihomotopy and d-homotopy

3.2.1. Definitions. It was mentioned in the introduction, Section 2.1.1, that two directed execution paths that can be deformed into each other along a one parameter deformation will yield the same result; a combinatorial explanation will be given in Section 3.2.2 below. It needs an adapted notion of *homotopy* to seriously involve methods of algebraic topology with directed spaces.

To this end, we need first to describe two directed intervals, both with $I = [0, 1]$ as the underlying topological space:

I : with $\vec{P}(I)$ consisting of the constant paths only;

\vec{I} : with $\vec{P}(\vec{I})$ consisting of all *non-decreasing* continuous paths $\varphi : I \rightarrow I$.

D- and dihomotopies, cf Grandis and Fajstrup et al [Gra03a, FGR06], from a d-space X to a d-space Y are homotopies that preserve certain d-space structures:

- DEFINITION 3.2.1. (1) A *dihomotopy* is a d-map $H : X \times I \rightarrow Y$. The d-maps H_0, H_1 are called *dihomotopic*: $H_0 \simeq H_1$.
- (2) A *d-homotopy* is a d-map $H : X \times \vec{I} \rightarrow Y$ establishing a relation $H_0 \preceq H_1$ (from H_0 to H_1).
- (3) Two d-maps $f, g : X \rightarrow Y$ are called *d-homotopic* if there is a finite sequence $f \preceq f_1 \succeq f_2 \preceq f_3 \dots f_n = g$ of “zig-zag” d-homotopies connecting them.

Obviously, d-homotopy implies dihomotopy.

There are also pointed and relative versions of these definitions. A particularly relevant case concerns directed homotopy of directed paths ($X = I$) with given end points.

3.2.2. Motivation. Here is how *directed homotopy* relates to the combinatorial framework of HDAs: The 1-skeleton of an HDA can be viewed as a (directed) graph. Directed paths on the 1-skeleton model executions that are locally sequential. Two paths p, q on the 1-skeleton with the same end-points are *elementarily*

equivalent if there are decompositions $p = p_- * p_0 * p_+$, $q = p_- * q_0 * p_+$ (same prefix and same postfix) and p_0, q_0 are directed paths (with the same end points) on the 1-skeleton of the same cube within the HDA¹; this defines a reflexive and symmetric relation on all such directed paths. By construction of the HDA (cf Section 3.2.1), the paths p_0 and q_0 will always yield the same result for a computation along the respective schedules; hence, so do p and q .

Combinatorial dihomotopy is the equivalence relation obtained by taking the transitive closure of elementary equivalence defined above. It is a direct consequence that executions (computations) that correspond to combinatorially dihomotopic directed paths on the 1-skeleton (and also what may arise from local interleavings) will yield the *same* result; non-dihomotopic paths *may* have different results.

General d-homotopy and dihomotopy are infinitesimal versions of combinatorial homotopy of directed paths (in a subdivided model space). For this motivation, it does not play a role which of the two relations one considers: Any two paths p, q on a k -cube from the bottom to the top are both d-homotopic and dihomotopic: Consider the path $p \vee q : I \rightarrow \square_k$, $(p \vee q)(t) = p(t) \vee q(t)$. Here \vee denotes the componentwise maximum. Both p and q are connected to $p \vee q$ by a linear d-homotopy: $p, q \preceq p \vee q$, and hence $p \preceq p \vee q \succeq q$.

3.2.3. Dihomotopy versus homotopy of directed paths. Two directed paths that are dihomotopic (d-homotopic) are certainly homotopic - forgetting about order(s) on the deformation interval. The reverse is in general *not* true. Simple examples are the “room with three barriers”, cf Fajstrup et al. [FGR06], or a 3D-progress graph in the form of a cube from which 4 hyperrectangles (forming pairs of corner wedges) have been removed, cf Raussen [Rau06] and Figure 3.1.

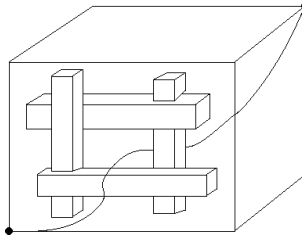


FIGURE 3.1. A d-path that is homotopic but not dihomotopic to a d-path on the boundary

In both cases, it is easy to convince yourself that certain d-paths are homotopic but not dihomotopic to each other. A formal proof is more difficult; it can best be achieved by abstract methods discussed in Chapter 7.

REMARK 3.2.2. For HDA (cubical complexes) of *dimension 2*, it was shown in Raussen [Rau10] that homotopic d-paths are dihomotopic. Hence 2-dimensional complexes are very special and not very useful as background for intuition about results on HDA of higher dimensions.

¹Just for this purpose, one might restrict to 2-dimensional complexes, or to the 2-skeleton of the complex. But that restriction would often make algorithms less efficient!

Another essential difference between dihomotopy and homotopy concerns *cancellation*: A homotopy $p_1 * p_2 \simeq p_1 * p_3$ of paths rel end point implies the existence of a homotopy $p_2 \simeq p_3$. A similar statement is *not* true for dihomotopy. An easy counter-example is found on the 2D-complex that is the boundary of a 3D cube with the lower facet removed. D-paths from bottom to top are all dihomotopic, whereas there are non-dihomotopic paths on the boundary of the removed facet.

3.2.4. Dihomotopy versus d-homotopy. Of course, d-homotopy of d-maps implies dihomotopy. It seems to be an open question in which generality the reverse holds. Fajstrup [Faj05] showed that d-paths on the 1-skeleton of a cubical complex that are dihomotopic are necessarily also d-homotopic. In greater generality, a directed version of a simplicial/cubical approximation theorem has recently been proved by Krishnan [Kri13].

Dihomotopy does not always imply d-homotopy, not even for directed paths. Here is a simple (but non-cubical) example due to Fajstrup: Consider the directed suspension of an interval I with (only) *constant* d-paths; the d-paths on that suspension from the bottom to the top point are contained in subspaces $\{a\} \times I$ that a d-homotopy (but not a dihomotopy) needs to respect.

3.3. A first invariant: The fundamental category of a d-space

In classical algebraic topology, the space of all paths $P(X)(x_0; x_1)$ in X from x_0 to x_1 is homotopy equivalent to the loop space $\Omega(X)(x_0)$ if just x_0 and x_1 belong to the same path component. The easy construction of a homotopy equivalence makes essential use of the *reverse* of a path connecting x_0 and x_1 . A similar statement is wrong for d-paths in directed spaces; one needs to take end points (or more generally, end sets) as essential parts of any algebraic structure capturing directed homotopy.

On the level of directed paths, appropriate structures are given by the *path category* $\vec{P}(X)$ and the *fundamental category* $\vec{\pi}_1(X)$ of a directed space X .

DEFINITION 3.3.1. (Grandis [Gra03a, Gra09]) Both the path category and the fundamental category of a directed space X have the elements of X as objects. The morphisms from x_0 to x_1 are given by

- $\vec{P}(X)(x_0, x_1)$: the space of d-paths from x_0 to x_1 (with the topology inherited from the compact-open topology on the space of all paths)
- $\vec{\pi}_1(X)(x_0, x_1)$: the set of dihomotopy (d-homotopy) classes of d-paths from x_0 to x_1 .

Composition of morphisms in both categories is given by concatenation (of paths, respectively their di/d-homotopy classes). The path category is enriched in the category \mathcal{Top} .

In Chapter 6, we will also consider the trace category $\vec{T}(X)$ of X whose morphism are spaces of traces (reparametrization equivalence classes) of d-paths between given end points (with the quotient topology; also $\vec{T}(X)$ is \mathcal{Top} -enriched). There are obvious forgetful functors $\vec{P}(X) \rightarrow \vec{T}(X) \rightarrow \vec{\pi}_1(X)$.

The most important result at this level – making calculations possible at least in relatively easy cases – is a van Kampen type result for fundamental categories. Note that a d-subspace $X_i \subset X$ has d-paths $\vec{P}(X_i) = \vec{P}(X) \cap P(X_i)$.

THEOREM 3.3.2. (Grandis [Gra03a]) *Let X denote a d -space with two d -subspaces X_1 and X_2 ; let $X_0 = X_1 \cap X_2$. If $X = \text{int}X_1 \cup \text{int}X_2$, then the following diagram of fundamental categories (induced by inclusions) is a pushout diagram in Cat :*

$$\begin{array}{ccc} \vec{\pi}_1(X_0) & \longrightarrow & \vec{\pi}_1(X_1) \\ \downarrow & & \downarrow \\ \vec{\pi}_1(X_2) & \longrightarrow & \vec{\pi}_1(X) \end{array}$$

The proof (in Grandis [Gra03a, Gra09]) – along the lines of the proof of the classical van Kampen theorem – makes essential use of “zig-zag” d -homotopies. Hence, the morphisms in the fundamental category have to be considered as d -homotopy classes of d -paths.

3.4. Outlook and Discussion

It depends very much on the aim of a study which framework one should choose. If one wishes to study homotopy theoretic properties, then the categories of d -spaces (Grandis) or of streams (Krishnan) are to prefer; the categories of flows (Gaucher), cf Chapter 8, has similar aims.

Cubical complexes have the huge advantage of combining topological, combinatorial and directed structures in a very natural way. This is why concrete results and calculations have mainly been established for such spaces, cf Chapter 7. Simplicial complexes, cf Ziemiański [Zie12b] might also become useful. Lpo-spaces filter out some of the general properties of cubical complexes (local antisymmetry and the non-existence of vortices). It is true that they do not form well-behaved categories – but neither do differentiable manifolds!

Localization techniques and components

Given a d-space X . As explained in Chapter 3, it comes equipped with (topological) path spaces $\bar{P}(X)(x_0, x_1)$ and their sets of components $\bar{\pi}_1(X)(x_0, x_1)$ forming the morphisms of the path category, resp. the fundamental category of X . It is relevant to ask how sensitive these path spaces and their components are with respect to *variations of the end points* x_0 and x_1 . If homotopy types (or some algebraic invariants) only change at certain thresholds, one may compress the representation of these categories without losing information.

4.1. Weakly invertible morphisms and wide subcategories

4.1.1. Problem and aim: State space explosion problem and components.

Discrete models (transition systems etc.) in concurrency theory share the property that the number of discrete states grows very fast with respect to the length (or rather complexity) of each individual program and also with respect to the number of processors. This is known as the *state space explosion problem*. At first sight, this problem does not get any better by replacing discrete state spaces with infinite state spaces, as with Higher Dimensional Automata. On the other hand, the state spaces (in the form of cubical complexes) are well-structured, and hence, one may hope that they can be decomposed in such a way that the homotopy types of path spaces (or at least their components) only change along some very specific “cut-locuses”; this would mean, that the essential information in path or fundamental categories can be compressed to quotient categories of a much smaller size. It was the aim of the series of papers Goubault and Raussen [GR02], Raussen [Rau03], Fajstrup et al [FGHR04], later extended by Goubault and Haucourt [GH07], to achieve such a compression of information in the *fundamental category* of a d-space by using a *calculus of fractions*.

4.1.2. Weakly invertible morphisms. To achieve this, one identifies in the fundamental category $\bar{\pi}_1(X)$ a system (in fact a wide subcategory) \mathcal{W} of so-called *weakly invertible* morphisms. A morphism (d-path class) $\sigma \in \bar{\pi}_1(X)(x, y)$ is called *weakly invertible*¹ if (pre and post-) compositions with σ in form of the maps

$$\begin{array}{ccc} \sigma^\sharp : \bar{\pi}_1(X)(y, z) & \rightarrow & \bar{\pi}_1(X)(x, z); & \sigma_\sharp : \bar{\pi}_1(X)(v, x) & \rightarrow & \bar{\pi}_1(X)(v, y) \\ \alpha & \mapsto & \sigma * \alpha & \beta & \mapsto & \beta * \sigma \end{array}$$

are *bijections* – as soon as the domains are non-empty.

REMARK 4.1.1. (1) The notions discussed apply to general categories, not only to the fundamental category.

¹Goubault and Haucourt [GH07] use the term Yoneda morphism instead.

- (2) Weakly invertible morphisms can be interpreted to be those that do not contribute to a “decision” when concatenated with other morphisms. The aim is to take a quotient with respect to a subcategory of the weakly invertible morphisms, as large as possible.
- (3) In Section 5.1, we will formulate other versions of weak invertibility and also generalize the localization techniques below.

4.1.3. Ir and pure subcategories. In order to apply localization techniques and to arrive at quotient categories with nice properties, we need further assumptions to a wide subcategory $\Sigma \subseteq \mathcal{W}$:

DEFINITION 4.1.2. (Gabriel and Zisman [GZ67], Borceux [Bor94]):

- (1) Σ satisfies the *lr-extension* properties (admits a left/right calculus of fractions) if every diagram of morphisms in $\tilde{\pi}_1(X)$

$$\begin{array}{ccc}
 x' & \xrightarrow{\gamma'} & y' \\
 \sigma \uparrow & & \uparrow \sigma' \\
 x & \xrightarrow{\gamma} & y
 \end{array}
 \qquad
 \begin{array}{ccc}
 x' & \xrightarrow{\gamma} & y' \\
 \sigma' \uparrow & & \uparrow \sigma \\
 x & \xrightarrow{\gamma'} & y
 \end{array}$$

with $\sigma \in \Sigma$ can be filled in with $\sigma' \in \Sigma$ (and γ' in $\tilde{\pi}_1(X)$);

- (2) Σ is *pure* if it only allows decompositions within Σ , ie $\sigma_1 * \sigma_2 \in \Sigma$ implies $\sigma_1, \sigma_2 \in \Sigma$.

REMARK 4.1.3. Applied to weakly invertible morphisms, the condition in Definition 4.1.2(1) tells you that a “non-decision” at the start point can be reflected by a “non-decision” at the end point and vice versa. Purenness means that a decision cannot be converted to a non-decision by pre-or post-composition.

4.2. Localization, categories of fractions and component categories

4.2.1. The setup. The general idea, initially formulated in Goubault and Raussen [GR02] and in Raussen [Rau03], is to formally invert a subcategory of the weakly invertible morphisms (turning those into isomorphisms) and to consider a quotient category (turning isomorphisms into identities).

In general, given a wide subcategory $\Sigma \subset \mathcal{C}$ of a category \mathcal{C} containing the \mathcal{C} -isomorphisms, one may consider the category “of fractions” $\mathcal{C}[\Sigma^{-1}]$ in which a formal inverse σ^{-1} has been added to every morphism $\sigma \in \Sigma$. It is universal with respect to functors that send all Σ -morphisms into isomorphisms. To make the exposition here as simple as possible, we work with a pure wide subcategory Σ satisfying the lr-extension property right away:

It is easy to see that all morphisms of $\mathcal{C}[\Sigma^{-1}]$ then have a description of the form $\sigma^{-1} \circ \alpha$, resp. $\beta \circ \sigma^{-1}$ with $\sigma \in \Sigma$. A morphism of the form $\sigma_1^{-1} \circ \sigma_2$, resp. $\sigma_1 \circ \sigma_2^{-1}$ is invertible in $\mathcal{C}[\Sigma^{-1}]$; such a morphism is called a Σ -zig-zag.

Two objects x, y of \mathcal{C} are called Σ -equivalent ($x \simeq_{\Sigma} y$) if there exists a Σ -zig-zag-morphism between them. The equivalence classes with respect to that relation are called the Σ -components of \mathcal{C} ; they are the (usual) path components *with respect to the Σ -zig-zag morphisms*. Moreover, we generate an equivalence relation on the morphisms of $\mathcal{C}[\Sigma^{-1}]$ by requiring that $\tau \simeq \tau \circ \sigma$, $\tau \simeq \sigma \circ \tau$ whenever $\sigma \in \Sigma$ and the composition is defined.

The *component category* $\pi_0(\mathcal{C}; \Sigma)$ of the category \mathcal{C} with respect to Σ has as objects the Σ -components; the morphisms from $[x]$ to $[y]$ are the equivalence classes of morphisms in $\bigcup_{x' \simeq x, y' \simeq y} \mathcal{C}[\Sigma^{-1}]$. Two morphisms in $\pi_0(\mathcal{C}; \Sigma)$ that are represented by $\tau_i \in \mathcal{C}(x_i, y_i), 1 \leq i \leq 2$, with $y_1 \simeq_\Sigma x_2$ can be composed by inserting any Σ -zig-zag-morphism connecting y_1 and x_2 , cf. Fajstrup et al. [FGHR04] for details.

Taking equivalence classes results in a quotient functor $q_\Sigma : \mathcal{C} \rightarrow \pi_0(\mathcal{C}; \Sigma)$.

4.2.2. Properties of component categories. The most important properties of components and the quotient function q_Σ shown in Fajstrup et al. [FGHR04] can be summarized as follows:

PROPOSITION 4.2.1. (Fajstrup et al. [FGHR04, Proposition 2 – 7]) *Let \mathcal{C} denote a category that has only identities as endomorphisms. Let $\Sigma \subset \mathcal{C}$ denote a pure wide subcategory of weakly invertible morphisms satisfying the lr-extension properties,² cf. Definition 4.1.2.*

Let x, y, z denote objects in \mathcal{C} .

- (1) $\Sigma(x, y)$ is either empty or it consists of a single morphism (in the latter case x, y are Σ -equivalent).
- (2) If x and y are Σ -equivalent and $f \in \mathcal{C}(x, z), g \in \mathcal{C}(z, y)$, then $f, g \in \Sigma$ (and hence z is Σ -equivalent to x and y).
- (3) Let $x, y \in \mathcal{C}$ for a component $C \subseteq \text{ob}(\mathcal{C})$. Every morphism $\tau' \in \mathcal{C}(x', y')$ with $x' \in C$ (resp. $y' \in C$) is Σ -equivalent to a morphism $\tau \in \mathcal{C}(x, -)$ (resp. $\tau \in \mathcal{C}(-, y)$).
- (4) For every object $x \in C$, every morphism in $\pi_0(\mathcal{C}; \Sigma)(C, D)$ has a lift to a morphism in $\mathcal{C}(x, y)$ for some $y \in D$.
- (5) If, moreover, $\pi_0(\mathcal{C}; \Sigma)(C, D)$ is finite, then there exists $y \in D$ such that the quotient map $q_C : \mathcal{C}(x, y) \rightarrow \pi_0(\mathcal{C}; \Sigma)(C, D)$ is a bijection.
- (6) Every isomorphism in $\pi_0(\mathcal{C}; \Sigma)$ is an endomorphism.
- (7) If $\tau_1 \circ \tau_2 \in \pi_0(\mathcal{C}; \Sigma)(C, C)$ is an isomorphism, then the $\tau_i, 1 \leq i \leq 2$, are isomorphisms.
- (8) $\pi_0(\mathcal{C}; \Sigma)$ has only identities as endomorphisms.

REMARK 4.2.2. The lifting properties (3) and (4) show the usefulness of the construction to yield information concerning the original category. Properties (6) and (7) show that it is impossible to return to a component that has been left.

Under reasonable additional assumptions, cf Fajstrup et al [FGHR04, Section 5.3], we were able to show that the component category $\pi_0(\mathcal{C}; \Sigma)$ has desirable properties. In particular, cf [FGHR04, Proposition 9], one can define the concept of neighbouring Σ - components C_1, C_2 ; these allow *precisely one* morphism in $\pi_0(\mathcal{C}; \Sigma)(C_1, C_2)$.

4.2.3. Further developments. How to choose a convenient large subcategory of the weakly invertible morphisms, that is both pure and that satisfies the extension properties? It is not difficult to see that the subcategories of a given category that satisfy the extension properties form a lattice with a maximal element (Fajstrup et al [FGHR04, Lemma 5]); this is the largest subcategory satisfying the extension properties. But there seems to be no way to find a maximal *pure* subcategory of

²Not all conditions are needed in all statements below. For details, consult Fajstrup et al. [FGHR04].

a given system of morphisms, in general; let alone one that satisfies extension properties.

4.3. Outlook and discussion

This problem has been overcome by Goubault and Haucourt [GH07] by strengthening the extension properties from Definition 4.1.2:

DEFINITION 4.3.1. (Goubault and Haucourt [GH07]) A wide subcategory $\Sigma \subset \mathcal{C}$ satisfies the *strong* extension properties if the diagrams in Definition 4.1.2 can be filled in so that they yield pushout, resp. pullback squares in \mathcal{C} .

REMARK 4.3.2. This means that the pushout (resp. pullback) is universal with respect to all other fillouts. The intuitive idea is that the pushout arises from a \vee -, resp. \wedge -operation in a lattice.

As a result, Goubault and Haucourt show that the strong extension properties *imply* pureness, in the following sense:

PROPOSITION 4.3.3. (Goubault and Haucourt [GH07]) *Let \mathcal{B} denote a wide subcategory of \mathcal{C} and suppose that the pair $(\mathcal{C}, \text{Iso}(\mathcal{C}))$ is pure.*

- (1) *If $(\mathcal{C}, \mathcal{B})$ satisfies the strong extension properties, then $(\mathcal{C}, \mathcal{B})$ is pure.*
- (2) *The family of all wide subcategories $\text{Iso}(\mathcal{C}) \subseteq \mathcal{D} \subseteq \mathcal{B} \subseteq \mathcal{C}$ such that $(\mathcal{C}, \mathcal{D})$ satisfies the strong extension properties is a complete lattice; in particular, there is a wide subcategory $\Sigma_{\mathcal{B}} \subseteq \mathcal{B}$ such that $(\mathcal{C}, \Sigma_{\mathcal{B}})$ satisfies the strong extension properties and such that $\mathcal{D} \subseteq \Sigma_{\mathcal{B}}$ for all \mathcal{D} above.*

Hence, this maximal subcategory satisfying the strong extension properties is a good candidate for the definition of components in *loopfree* categories.

Moreover, also in Goubault and Haucourt [GH07], the authors establish an equivalence between the category of fractions $\mathcal{C}[\Sigma^{-1}]$ and the quotient category \mathcal{C}/Σ using generalized equivalences that had previously been investigated by Bednarczyk et al [BBP99]. They go on to show that the van Kampen theorem for the fundamental category from Grandis [Gra03a] infers a similar statement for component categories (with respect to a maximal subcategory of the weakly invertible morphisms satisfying the strong extension properties).

For many applications, it may only be important to distinguish elements of a d-space (and d-paths starting at such elements) with respect to their *future*. In that case, only r-extensions properties are relevant. A framework to handle future components in that direction has been dealt with in Goubault et al [GHK10].

A general categorical approach to invariants of directed spaces

5.1. Categorical approaches

How about algebraic topological invariants of a d-space X ? Of course, one may define invariants of path spaces $\vec{P}(X)(x, y)$ or their quotient trace spaces $\vec{T}(X)(x, y)$ (up to reparametrization, cf Chapter 6) for given $x, y \in X$ and have them organized by way of various categories related to X itself. In fact, since both source and target play a role, it is more natural to use categories related to the product $X \times X$ for indexing purposes; this is certainly necessary for categories that are not acyclic – in the notation of Kozlov [Koz08] – or loop-free – in the notation of Haucourt [Hau06]. This bookkeeping point of view and also an analysis of associated component categories (with distinctions up to various invariants from algebraic topology) are the main contributions of Raussen [Rau07]:

5.1.1. Preorder categories as indexing categories and functors. A d-space X comes equipped with a natural preorder $x \preceq y \Leftrightarrow \vec{P}(X)(x, y) \neq \emptyset$. For all the preorder categories below, the *objects* are the pairs $(x, y) \in X \times X$ with $x \preceq y$.

The *morphisms* in the category $\vec{D}(X)$ are $\vec{D}(X)((x, y), (x', y')) := \vec{T}(X)(x', x) \times \vec{T}(X)(y, y')$ with composition given by pairwise contra-, resp. covariant concatenation. Hence, $\vec{D}(X)$ is a full subcategory of the category $\vec{T}(X)^{op} \times \vec{T}(X)$. Note that every morphism $(\sigma_x, \sigma_y) \in \vec{T}(X)(x', x) \times \vec{T}(X)(y, y')$ decomposes as

$$(\sigma_x, \sigma_y) = (c_{x'}, \sigma_y) \circ (\sigma_x, c_y) = (\sigma_x, c_{y'}) \circ (c_x, \sigma_y)$$

with $c_u \in \vec{T}(X)(u, u)$ the constant trace at $u \in X$.

Trace spaces with varying pairs of end points in X can be organised by the trace space functor $\vec{T}^X : \vec{D}(X) \rightarrow \text{Top}$ given by $\vec{T}^X(x, y) = \vec{T}(X)(x, y)$ and $\vec{T}^X(\sigma_x, \sigma_y)(\sigma) := \sigma_x \circ \sigma \circ \sigma_y \in \vec{T}(X)(x', y')$ for $\sigma \in \vec{T}(X)(x, y)$. This functor can be viewed as (a restriction of) the *Top*-enriched hom-functor of $\vec{T}(X)$.

A d-map $f : X \rightarrow Y$ induces a functor $\vec{D}(f) : \vec{D}(X) \rightarrow \vec{D}(Y)$ with $\vec{D}(f)(x, y) = (fx, fy)$ and $\vec{D}(f)(\sigma_x, \sigma_y) = (\vec{T}(f)(\sigma_x), \vec{T}(f)(\sigma_y)) = (f \circ \sigma_x, f \circ \sigma_y)$; moreover, it induces a natural transformation $\vec{T}(f)$ from \vec{T}^X to \vec{T}^Y .

A homotopical variant is given by the category $\vec{D}_\pi(X)$ with the same objects as above and with $\vec{D}_\pi(X)((x, y), (x', y')) := \vec{\pi}_1(X)(x', x) \times \vec{\pi}_1(X)(y, y')$. Hence, this category is a full subcategory of the category $\vec{\pi}_1(X)^{op} \times \vec{\pi}_1(X)$, and $\vec{\pi}_1(X)$ denotes the fundamental category (cf Sect. 3.3). It comes with a functor $\vec{T}_\pi^X : \vec{D}_\pi(X) \rightarrow \text{Ho-Top}$ into the homotopy category; a d-map $f : X \rightarrow Y$ induces a natural transformation $\vec{T}_\pi(f)$ from \vec{T}_π^X to \vec{T}_π^Y . Together with the (vertical)

forgetful functors, we obtain a commutative diagram

$$(5.1) \quad \begin{array}{ccc} \vec{D}(X) & \xrightarrow{\vec{T}^X} & Top \\ \downarrow & & \downarrow \\ \vec{D}_\pi(X) & \xrightarrow{\vec{T}_\pi^X} & Ho - Top. \end{array}$$

The functors \vec{T}^X and \vec{T}_π^X may be composed with homology functors into categories of (graded) abelian groups, R -modules or graded rings. In particular, we obtain, for $n \geq 0$, functors $\vec{H}_{n+1}(X) : \vec{D}(X) \rightarrow Ab$ with $(x, y) \mapsto H_n(\vec{T}(X)(x, y))$ and $(\sigma_x, \sigma_y)_*$ given by the map induced on n -th homology groups by concatenation with those two traces on trace space $\vec{T}(X)(x, y)$. This functor factors over $\vec{D}_\pi(X)$ by homotopy invariance. In the same spirit, one can define homology with coefficients and cohomology. A d-map $f : X \rightarrow Y$ induces a natural transformation $\vec{H}_{n+1}(f) : \vec{H}_{n+1}(X) \rightarrow \vec{H}_{n+1}(Y)$, $n \geq 0$.

Composing with the functor $\pi_0 : Top \rightarrow Sets$ that associates to a topological space its set of path components, defines a functor $\vec{\Pi}_1 : \vec{D}(X) \rightarrow Sets$ with $\vec{\Pi}_1(X)(x, y) = \vec{\pi}_1(X)(x, y)$, the set of morphisms in the fundamental category – with dihomotopy and not d-homotopy, cf Section 3.2.4, as equivalence relation.

If one needs to take care of base points (essential for homotopy groups of the very often non-connected spaces of d-paths), more care is needed. For that purpose, one may use a factorization category (cf Baues [Bau89]) of the trace category; for details, cf Raussen [Rau07, Section 3.3].

5.1.2. Components with respect to functors. Using functors – like the ones discussed above – on preorder categories, one may define components as subspaces of $X \times X$, for a variety of algebraic topological invariants. In greater generality:

Consider a functor $F : \mathcal{C} \rightarrow \mathcal{D}$ between two small categories. A morphism $\sigma \in \mathcal{C}(x, y)$ will be called *F-invertible* if and only if $F(\sigma) \in \mathcal{D}(Fx, Fy)$ is a *D-isomorphism*. Let $\mathcal{C}_F(x, y) \subseteq \mathcal{C}(x, y)$ denote the set of all *F-invertible* morphisms from x to y . The collection of all $\mathcal{C}_F(x, y)$ form a wide subcategory \mathcal{C}_F of \mathcal{C} since the composition of two *F-invertible* morphisms obviously is *F-invertible* again; remark that $\mathcal{C}_F(x, y)$ contains the \mathcal{C} -isomorphisms.

For example, consider the functor $\vec{T}_\pi^X : \vec{D}_\pi(X) \rightarrow Ho - Top$ or the functors $\vec{H}_{n+1}(X) : \mathcal{D}_\pi(X) \rightarrow Ab$ from Sect. 5.1.1. A morphism $(\sigma_x, \sigma_y) \in \vec{D}_\pi(X)((x, y), (x', y'))$ is \vec{T}^X -invertible if and only if $\vec{T}(X)(\sigma_x, \sigma_y) : \vec{T}(X)(x, y) \rightarrow \vec{T}(X)(x', y')$ is a homotopy equivalence; it is \vec{H}_{n+1} -invertible if $(\sigma_x, \sigma_y)_* : H_n(\vec{T}(X)(x, y)) \rightarrow H_n(\vec{T}(X)(x', y'))$ is an isomorphism.

When \mathcal{C} is the homotopy preorder category $\vec{D}_\pi(X)$ and F one of the functors from Section 5.1.1, it makes sense to apply the framework of component categories from Section 4.2. Reasonable components can then be defined at least in the case when the \vec{T}^X -invertible morphisms satisfy the strong extension properties, cf. Definition 4.3.1. For details, compare Raussen [Rau07, Section 4].

5.2. Homotopy flows and d-homotopy equivalences

We discuss a candidate for a definition of the notion *directed homotopy equivalence* and an investigation of its properties:

5.2.1. Introduction. Which requirements should a d-map $f : X \rightarrow Y$ satisfy in order to qualify as a directed homotopy equivalence? Obviously, there should be a reverse d-map $g : Y \rightarrow X$ such that both $g \circ f$ and $f \circ g$ are d-homotopic to the respective identity maps. But this is not enough: The (d-path) structures on X and Y ought to be homotopically related, i.e., the maps $\tilde{T}(f) : \tilde{T}(X)(x, y) \rightarrow \tilde{T}(Y)(fx, fy)$ should be ordinary homotopy equivalences – for all x, y with $\tilde{T}(X)(x, y) \neq \emptyset$ – and that in a natural way. Moreover note the following:

EXAMPLE 5.2.1. The subspace $L = [0, 1] \times \{0\} \cup \{0\} \times [0, 1] \subset \mathbb{R}^2$ – the branch figure “letter L” – is homotopy equivalent to the one point space $O = \{(0, 0)\}$ included in it; there is a d-homotopy of the map $c \circ i$ (i : inclusion, c : the (constant) reverse map) to the identity map on L . Moreover, all non-empty trace spaces are contractible. But a branch should not be d-homotopy equivalent to a point!

5.2.2. Homotopy flows. The main tool in the definition below is the notion of a homotopy flow generalising the concept of a flow on a differentiable manifold. This notion will be used as an ingredient in the requirements for a d-homotopy equivalence. Moreover, it is useful in order to generate subcategories of weakly invertible d-paths (cf Section 4.1) and thus to reason about component categories (cf Section 4.2).

DEFINITION 5.2.2. (1) A d-map $H : X \times \vec{I} \rightarrow X$ is called a *future homotopy flow* if $H_0 = id_X$ and a *past homotopy flow* if $H_1 = id_X$.

(2) The sets consisting of all future homotopy flows, resp. of all past homotopy flows will be denoted by $\vec{P}_+C(X)$, resp. by $\vec{P}_-C(X)$.

Remark that we do not require that the maps $H(-, t) : X \rightarrow X$ are homeomorphisms.

The orbits of a flow have the following counterpart: For every $x \in X$, the map $H_x : \vec{I} \rightarrow X$, $t \mapsto H(x, t)$, is a d-path (with $H_x(0) = x$, resp. $H_x(1) = x$). Evaluation at $x \in X$ sends x to H_x and defines maps $H \mapsto H_x$

$$(5.2) \quad ev_+^x : \vec{P}_+C_0(X) \rightarrow \vec{T}(X)(x, -), \text{ resp. } ev_-^x : \vec{P}_-C_0(X) \rightarrow \vec{T}(X)(-, x)$$

by $H \mapsto H_x$. Every homotopy flow gives thus rise to a well-organized collection of d-paths.

Remark that a maximal element $x_+ \in X$ – the only d-path with source x_+ is constant – will be fixed under a future homotopy flow; likewise a minimal element under a past homotopy flow. Moreover, a branch point like in Example 5.2.1 is fixed under all (future, resp. past) homotopy flows.

Homotopy flows can be concatenated, for future homotopy flows as follows:

$$(H_1 * H_2)(x, t) = \begin{cases} H_1(x, 2t), & t \leq \frac{1}{2} \\ H_2(H_1(x, 1), 2t - 1), & t \geq \frac{1}{2}. \end{cases}$$

A homotopy flow $H_+ : X \times \vec{I} \rightarrow X$ and its restrictions $H_+^s : X \times \vec{I} \rightarrow X$, $H_+^s(x, t) = H_+(x, st)$ induce interesting maps on trace spaces, collected in the

homotopy commutative diagram below (similarly for a past homotopy flow H_-):

$$(5.3) \quad \begin{array}{ccc} \vec{T}(X)(x, y) & \xrightarrow{\vec{T}^X(c_x, H_{+y}^s)} & \vec{T}(X)(x, H_+(y, s)) \\ \downarrow \vec{T}(H_+(-, s)) & & \uparrow \vec{T}^X(H_{+x}^s, c_{H_+(y, s)}) \\ \vec{T}(X)(H_+(x, s), H_+(y, s)) & & \end{array}$$

A homotopy flow on X does not change the topology of trace spaces if it induces homotopy equivalences on associated trace spaces:

- DEFINITION 5.2.3. (1) A future homotopy flow $H : X \times \vec{I} \rightarrow X$ is called *automorphic* if, for all $x, y \in X$ with $\vec{T}(X)(x, y) \neq \emptyset$ and all $s \in I$, the map $\vec{T}(H_+^s)$, (vertical in (5.3)) is a *homotopy equivalence*. Similarly for past a past homotopy flow and $\vec{T}(H_-^s)$.
- (2) A self-d-map $f : X \rightarrow X$ is called a future/past-automorphism if there exists an automorphic future/past homotopy flow connecting f and the identity on X .

In particular, given such an automorphic homotopy flow H_+ , resp. H_- , the maps $f = H_+(-, 1)$, resp. $g = H_-(-, 0)$ induce homotopy equivalences

$$\vec{T}(f) : \vec{T}(X)(x, y) \rightarrow \vec{T}(X)(fx, fy) \text{ resp. } \vec{T}(g) : \vec{T}(X)(gx, gy) \rightarrow \vec{T}(X)(x, y).$$

Relations to the remaining maps on trace spaces in (5.3) are formulated in the easy

LEMMA 5.2.4. *Let H denote a future/past homotopy flow on X .*

- (1) *If all concatenation maps (the skew ones in (5.3)) are homotopy equivalences, then H is automorphic.*
- (2) *Let H be automorphic. If one of the concatenation maps (skew in (5.3)) is a homotopy equivalence, then the other is as well.*
- (3) *$P_+C(X)$ satisfies the r -extension properties, and $P_-C(X)$ satisfies the l -extension property.*

Weaker properties (concerning maps on trace spaces that induce isomorphisms on homotopy, resp. homology groups) are also formulated in Raussen [Rau07, Definition 5.9].

5.2.3. d-homotopy equivalences: Definition and properties. The following is a suggestion for a reasonable definition of a d-homotopy equivalence making sure that trace spaces are related by homotopy equivalences. The conditions should be seen as requirements to a d-homotopy *class* of d-maps from X to Y :

- DEFINITION 5.2.5. (1) A d-map $f : X \rightarrow Y$ is called a *future d-homotopy equivalence* if there exist d-maps $f_+ : X \rightarrow Y, g_+ : Y \rightarrow X$ such that f, f_+ are d-homotopic and *automorphic* d-homotopies $H^X : id_X \rightarrow g_+ \circ f_+$ on X and $H^Y : id_Y \rightarrow f_+ \circ g_+$ on Y (ie $H^X : X \times \vec{I}$ with $H_0^X = id_X$ and $H_1^X = g_+ \circ f_+$) etc.

- (2) The d-map $f : X \rightarrow Y$ is called a *past d-homotopy equivalence* if there exist d-maps $f_- : X \rightarrow Y, g_- : Y \rightarrow X$ such that f, f_- are d-homotopic and *automorphic* d-homotopies $H^X : g_- \circ f_- \rightarrow id_X$ on X and $H^Y : f_- \circ g_- \rightarrow id_Y$ on Y .
- (3) A d-map f is called a *d-homotopy equivalence* if it is both a future and a past d-homotopy equivalence.

Remark that a self-d-homotopy equivalence preserves branch points (like in Example 5.2.1).

d-homotopy equivalences have the desired properties:

PROPOSITION 5.2.6. *The natural transformation $\tilde{T}_\pi(f) : \tilde{T}_\pi^X \rightarrow \tilde{T}_\pi^Y$ induced by a (past or future) d-homotopy equivalence $f : X \rightarrow Y$ between d-spaces X and Y is an equivalence, i.e., the induced maps $\tilde{T}(f)(x, y) : \tilde{T}(X)(x, y) \rightarrow \tilde{T}(Y)(fx, fy)$ are homotopy equivalences for all $x, y \in X$ with $x \preceq y$.*

PROOF. By abuse of notation, we write f, g instead of f^+, g^+ , resp. f^-, g^- in the following. In the diagram

$$\begin{array}{ccccccc} \tilde{T}(X)(x, y) & \xrightarrow{\tilde{T}(f)} & \tilde{T}(Y)(fx, fy) & \xrightarrow{\tilde{T}(g)} & \tilde{T}(X)(gfx, gfy) & \xrightarrow{\tilde{T}(f)} & \tilde{T}(Y)(fgfx, fgfy), \\ & & \swarrow I & & \searrow J & & \\ & & & & & & \end{array}$$

let I denote a homotopy inverse to $\tilde{T}(g) \circ \tilde{T}(f)$ and let J denote a homotopy inverse to $\tilde{T}(f) \circ \tilde{T}(g)$. Then $\tilde{T}(g)$ has a homotopy right inverse $\tilde{T}(f) \circ I$ and a homotopy left inverse $J \circ \tilde{T}(f)$. By general nonsense, the right homotopy inverse and the left homotopy inverse are homotopic to each other, and thus $\tilde{T}(g)$ is a homotopy equivalence. Since $\tilde{T}(g \circ f) = \tilde{T}(g) \circ \tilde{T}(f)$ is a homotopy equivalence by definition, the map $\tilde{T}(f)$ is a homotopy equivalence, as well. \square

Furthermore, we prove in [Rau07, Proposition 6.8] that future and past dihomotopy equivalences behave well under composition:

PROPOSITION 5.2.7. *The composition $g \circ f : X \rightarrow Z$ of (future or past) dihomotopy equivalences $X \xrightarrow{f} Y \xrightarrow{g} Z$ is again a (f/p) dihomotopy equivalence.*

5.3. Outlook and discussion

5.3.1. Construction of homotopy flows and of components? While the definition of a d-homotopy equivalence ensures some of the most desirable properties, it seems not that easy to construct the homotopy flows needed in the definition in a systematic way. The fact that they have to preserve branch points (for a more precise definition cf Raussen [Rau12b] or Section 7.3.1) and thus also the regions between branch points requires them to be very “stiff”. On the other hand, this is perhaps good news for handling components in simple cases, like for the PV-models from Section 2.1 and Chapter 7.

5.3.2. Topology change and relations to multidimensional persistence. It is clear that the discussion of topology change of trace spaces under variation of end points and the identification of components cannot be easy in general. In fact, variation of end points can be thought of as a (double) filtration on the trace spaces: You begin with trace spaces $\tilde{T}(X)(x, x)$ – trivial in spaces without directed loops

– and let start and end point drift further and further away from each other and observe topology changes – at certain thresholds – on the way. At a first glance, the situation resembles that investigated in the theory of (homological) persistence:

For a one-dimensional filtration with coefficients over a field k , the homology of a filtered space can be analyzed via the ranks of homology maps in the form of so-called barcodes, cf eg Edelsbrunner and Harer [EH10]. The theoretical background herefore is based on the classification of modules over the polynomial ring $k[t]$, cf Zomorodain and Carlsson [ZC05].

For a multi-scale filtration, the situation is much more complicated; one needs to analyze modules over a polynomial ring of several variables. In general, discrete invariants like the barcodes are not sufficient; compare Carlsson and Zomorodian [CZ09].

Our case is even more difficult; for at least three reasons:

- The state space has “holes” in cases of interest; a filtration is thus not homogeneous and theoretically understandable by considering a polynomial ring.
- There may and will often be more than one map comparing trace spaces between given pairs of end points (this is the essence of the category $D(X)$ and also $D_\pi(X)$).
- In full generality, one would have to consider a double (multi-scale) filtration, taking into accounts both end points.

Topology of executions: General properties of path and trace spaces

One of the fundamental tasks in directed algebraic topology is to translate information about a directed space (d-space) into information about the *space of directed paths* (d-paths, executions) in that space. To this end, one has to structure the path spaces and

- to define them as (ordinary) *topological* spaces and to study properties of such spaces,
- to define and study appropriate subspaces (eg subspaces of d-paths with given end points) and natural maps between such,
- to investigate the influence of (weakly increasing) reparametrizations and to study properties of quotient spaces (trace spaces).

We begin here with the last issue:

6.1. Reparametrizations and traces

6.1.1. In ordinary spaces – without directions. Before turning to spaces of d-paths, it turned out to be beneficial to study the space of paths $P(X) = X^I$ from an interval I into a Hausdorff space X in the compact-open topology more closely. This Section reports briefly on the work of Fahrenberg and Raussen [FR07] with a minor correction that appeared in Raussen [Rau09a]:

In differential geometry, one studies usually (spaces of) *regular* paths in a manifold. Those have never-vanishing speed, can be reparametrized by a diffeomorphism of the unit interval to yield a path parametrized by arc length – with *unit speed*.

The closest analogon for paths in a Hausdorff space leads to

DEFINITION 6.1.1. Let $p : I \rightarrow X$ denote a continuous path in a Hausdorff space X .

- (1) A path $p : I \rightarrow X$ is called *regular* if, for every interval $J \subseteq I$ with $p|_J$ constant, J is either a degenerate (one-point) interval $[a, a]$ or $J = I$.
- (2) A continuous map $\phi : I \rightarrow I$ is called a *reparametrization* if $\phi(0) = 0$, $\phi(1) = 1$ and if ϕ is *increasing*, i.e. if $s \leq t \in I$ implies $\phi(s) \leq \phi(t)$.
The subspace (and monoid under composition) of all reparametrizations (within the mapping space I^I) is denoted $\text{Rep}_+(I)$.
- (3) The subspace (and group under composition) of all (increasing) *homeomorphisms* within $\text{Rep}(I)$ is called $\text{Homeo}_+(I)$.
- (4) Two paths $p, q : I \rightarrow X$ are called *reparametrization equivalent* if there exist reparametrizations ϕ, ψ such that $p \circ \phi = q \circ \psi$.

It turns out that reparametrization equivalence is in fact an equivalence relation, but the proof of the transitivity property is non-trivial. The equivalence classes with respect to reparametrization equivalence are called *traces* in the space X , and the quotient space $T(X) = X^I /_{\text{Rep}_+(I)}$ of $P(X) = X^I$ is called the *trace space* for X . Likewise, one may consider the quotient space $T_R(X) = R(X) /_{\text{Homeo}_+(I)}$ of *regular traces*.

Since one can contract paths to their start point, trace spaces as such are not interesting from the homotopy point of view. This changes, when one restricts end points and looks at subspaces of all (regular) paths $R(X)(x_0, x_1) \subset P(X)(x_0, x_1)$ with given end points $x_0, x_1 \in X$. In the non-directed case, one may restrict attention to *loops* ($x_0 = x_1$) in every path-connected component, and it is then not difficult to see using path fibrations – and proved in Fahrenberg and Raussen [FR07, Remark 3.10]:

PROPOSITION 6.1.2. *For two elements $x_0, x_1 \in X$ in a Hausdorff space X , the inclusion map $R(X)(x_0, x_1) \hookrightarrow P(X)(x_0, x_1)$ is a weak homotopy equivalence.*

It is natural to ask whether every path is reparametrization equivalent to a regular path. Since a general path can have complicated (Cantor) sets of intervals on which it is constant, this is not clear right away. In fact, reparametrizations of the interval have an interesting algebraic and combinatorial structure investigated in Fahrenberg and Raussen [FR07, Section2]. Using these insights, we proved in Theorem 3.6 of that article:

THEOREM 6.1.3. *For every two elements $x_0, x_1 \in X$ of a Hausdorff space X , the map $i : T_R(X)(x_0, x_1) \rightarrow T(X)(x_0, x_1)$ is a homeomorphism.*

The proof of this theorem is surprisingly intricate and needs a thorough study of the monoid $\text{Rep}_+(I)$ of reparametrizations, a characterization of reparametrization by sets of *stop intervals* and relations between those.

REMARK 6.1.4. It seems to be difficult to prove statements about the quotient maps q and q_R in the diagram

$$\begin{array}{ccc} R(X)(x_0, x_1) & \xrightarrow{\subset} & P(X)(x_0, x_1) \\ \downarrow q_R & & \downarrow q \\ T_R(X)(x_0, x_1) & \xrightarrow[i]{\cong} & T(X)(x_0, x_1) \end{array}$$

in full generality. The map q_R is a quotient map for a free action of the contractible group $\text{Homeo}_+(I)$. But it is not clear whether this map is a fibration in general; one cannot apply the Vietoris-Begle theorem (cf Smale [Sma57]) either since the fibre $\text{Homeo}_+(I)$ is not compact.

If the map q_R is a fibration (with contractible fibers of type $\text{Homeo}_+(I)$), then it is a weak homotopy equivalence, and then q is so as well. This is certainly the case when there is a (“unit speed”) section of q_R leading to product decompositions $R(X)(x_0, x_1) \cong T_R(X)(x_0, x_1) \times \text{Homeo}_+(I)$ and $P(X)(x_0, x_1) \cong T(X)(x_0, x_1) \times \text{Rep}_+(I)$. In this case, both q_R and q are actually homotopy equivalences. For a framework where this occurs naturally, cf Section 6.2.

6.1.2. In directed spaces. Under mild extra assumptions, a variant of Theorem 6.1.3 holds for a d-space X , as well. Let $\vec{R}(X) = R(X) \cap \vec{T}(X)$ consist of the regular d-paths. The free action of the contractible group $\text{Homeo}_+(X)$ restricts to $\vec{R}(X)$ and yields the quotient space $\vec{T}_R(X) = \vec{R}(X)/_{\text{Homeo}_+(I)}$. We need to consider so-called *saturated* d-spaces:

DEFINITION 6.1.5. (Fahrenberg and Raussen [FR07]) A d-space X is called a *saturated* d-space if the underlying topological space is Hausdorff and satisfies the following additional property:

$$p \in \vec{P}(X), \varphi \in \text{Rep}_+(I) \text{ and } p \circ \varphi \in \vec{P}(X) \Rightarrow p \in \vec{P}(X).$$

This means, that if p becomes a d-path after a reparametrization, then it has to be a d-path itself. It is easy to saturate a given d-space to yield a d-space with possible additional d-paths; therefore it is no harm to assume that a d-space is saturated right away.

Remark that, unlike in the classical case, the topology of the spaces $\vec{T}(X)(x_0, x_1)$ will usually depend crucially on the choice of end points.

COROLLARY 6.1.6. For every two elements $x_0, x_1 \in X$ of a saturated d-space X , the map $i : \vec{T}_R(X)(x_0, x_1) \rightarrow \vec{T}(X)(x_0, x_1)$ is a homeomorphism.

It is not clear, in general, that the inclusion map $\vec{R}(X)(x_0, x_1) \hookrightarrow \vec{P}(X)(x_0, x_1)$ is a homotopy equivalence. But again, as in Section 6.1.1, if inclusion has a (“unit speed”) section, then $\vec{P}(X)(x_0, x_1) \cong \vec{R}(X)(x_0, x_1) \times \text{Rep}_+(I)$ and hence inclusion is a homotopy equivalence. Section 6.2 handles a case where this is naturally the case.

REMARK 6.1.7. K. Ziemiański [Zie12b, Section 5] has noted that spaces of d-paths and of traces in a d-space are homotopy equivalent if one just has a (length) function $l : \vec{P}(X) \rightarrow \mathbf{R}$ that is a homomorphism with respect to concatenation and addition, trivial only on constant dipaths, invariant under reparametrization and, moreover and crucially, *continuous*.

6.2. Trace spaces in cubical complexes

Our aim is to describe additional tools that allow a closer investigation of trace spaces $\vec{T}(X)(x_0, x_1)$ for a convenient d-space X .

Convenience has two aspects:

- There should be enough structure on the d-space to make sure that it is possible to describe an associated trace space in terms of a *finite complex*.
- Important classes of d-spaces arising as models for concurrency should be included.

The cubical complexes and their geometric realizations explained in Section 3.1.4 are candidates for convenient d-spaces, certainly satisfying the second criterion as models for Higher Dimensional Automata. With the d-space structure explained in Section 3.1.4, we shall explain that associated trace spaces enjoy several properties that pave the way for more detailed investigations. In short, it is shown in Raussen [Rau09b], that trace spaces in a cubical complex – under mild additional assumptions – are separable metric spaces which are locally contractible and locally compact. Moreover, it turns out that they have the homotopy type of

a CW-complex, and hence determination of topological invariants comes within reach.

6.2.1. Arc length parametrization and consequences. The L_1 -arc length of a d-path in a cubical complex was introduced and studied in Raussen [Rau09b, Rau12b]. The *signed* L_1 -length $l_1^\pm(p)$ of a path $p : I \rightarrow \square_n$ within a cube \square_n is defined as $l_1^\pm(p) = \sum_{j=1}^n p_j(1) - p_j(0)$. For any path p , that is the concatenation of finitely many paths each of which is contained in a single cube, the signed L_1 -length is defined as the sum of the lengths of the pieces. The result is independent of the choice of decomposition – and of the parametrization! Moreover, it is non-negative for every d-path and positive for every non-constant d-path.

This construction can be phrased more elegantly using differential one-forms on a cubical complex (a special case of the PL differential forms introduced by D. Sullivan [Sul77] in his approach to rational homotopy theory, or of the closed one-forms on topological spaces by M. Farber [Far02, Far04]): On an n -cube $e \simeq \square_n$, consider the particular 1-form $\omega_e = dx_1 + \cdots + dx_n \in \Omega^1(\square_n)$. It is obvious that $\omega_{\partial_i^k e} = (i_i^k)^* \omega_e$ with $i_i^k : |\partial_i^k e| \hookrightarrow |e|$ denoting inclusion. Pasting together, one arrives at a particular (closed!) 1-form ω_X on every pre-cubical set X – the one-form that reduces to ω_e on every cell e in X .

The signed length of a (piecewise differentiable) path γ on X can then be defined as $l_1^\pm(\gamma) = \int_0^1 \gamma^* \omega_X$ and extended to continuous paths using uniformly converging sequences of such piecewise differentiable paths. This length

- is invariant under orientation preserving reparametrization;
- changes sign under orientation reversing reparametrization;
- is additive under concatenation and non-negative for d-paths.
- yields a *continuous* map $l_1^\pm : P(X)(x_0, x_1) \rightarrow \mathbf{R}$.

An application of Stokes' theorem shows:

PROPOSITION 6.2.1. *Two paths $p_0, p_1 \in P(X)(x_0, x_1)$ that are homotopic rel end points have the same signed length: $l_1^\pm(p_0) = l_1^\pm(p_1)$.* \square

A more direct proof can be given along the lines of Raussen [Rau09b] using the continuous d-map $s : X \rightarrow S^1 = \mathbf{R}/\mathbf{Z}$ given by $s(e; x_1, \dots, x_n) = \sum x_i \bmod 1$. We think of S^1 as a pre-cubical set with one vertex and one edge from that vertex to itself. Then the map $l_1^\pm(S^1) : P(S^1) \rightarrow \mathbf{R}$ coincides with the map that associates to $p \in P(S^1)$ the real number $\hat{p}(1) - \hat{p}(0)$ for an arbitrary lift \hat{p} of p under the exponential map $\exp : \mathbf{R} \rightarrow S^1$. It follows from the definition that the arc length $l_1^\pm(X)$ factors for an arbitrary cubical complex X :

$$l_1^\pm(X) : P(X) \xrightarrow{s_\#} P(S^1) \xrightarrow{l_1^\pm(S^1)} \mathbf{R}$$

with the following consequences:

- PROPOSITION 6.2.2.**
- (1) *The function $l_1^\pm : P(X) \rightarrow \mathbf{R}$ is continuous.*
 - (2) *$l_1^\pm(p) \equiv s(p(1)) - s(p(0)) \bmod 1$ for $p \in P(X)$.*
 - (3) *$l_1^\pm(P(X)(x_0, x_1))$ is constant mod 1 for every pair of points $x_0, x_1 \in X$.*
 - (4) *l_1^\pm is constant on any connected component, i.e., on any homotopy class of paths in $P(X)(x_0, x_1)$. In particular, it induces a map $l_1^\pm : \pi_1(X)(x_0, x_1)$ into a coset in $\mathbf{R} \bmod \mathbf{Z}$.*

Similar results hold for spaces of *directed paths* $\vec{P}(X)(-, -)$ and of *traces* $\vec{T}(X)(-, -)$ (with *dihomotopy* classes corresponding to components).

REMARK 6.2.3. Ordinary Euclidean arc length (for piecewise differentiable functions) is *not* a continuous function on a path space. A family of graphs of oscillating functions with increasing frequency and decaying amplitude may have constant arc length and converge to the graph of a constant function with a smaller arc length.

We will now consider the restriction $l_1 : \vec{P}(X) \rightarrow \mathbf{R}_{\geq 0}$ of l_1^\pm to spaces of d-paths. We call a d-path parametrization $p : I \rightarrow X$ *natural* if $l_1(p|_{[0,t]}) = t \cdot l_1(p)$ for all $t \in I$; such a path moves at “unit speed” with respect to l_1 . Naturally parameterized d-paths form the subspace $\vec{N}(X) \subset \vec{P}(X)$.

PROPOSITION 6.2.4. *Let X denote a cubical complex; $x_0, x_1 \in X$.*

- (1) $\vec{N}(X)(x_0, x_1) \subset \vec{P}(X)(x_0, x_1)$ *is a deformation retract.*
- (2) *All maps in the diagram*

$$\begin{array}{ccc} \vec{N}(X)(x_0, x_1) & \xhookrightarrow{\quad} & \vec{P}(X)(x_0, x_1) \\ & \searrow & \swarrow \\ & \vec{T}(X)(x_0, x_1) & \end{array}$$

are homotopy equivalences.

PROOF. A homotopy inverse to inclusion is given by the naturalization map $\text{nat} : \vec{P}(X)(x_0, x_1) \rightarrow \vec{N}(X)(x_0, x_1)$, $\text{nat}(p)(t) = p(l_1^{-1}(t))$ – which is well-defined (!), continuous and homotopic to the identity in $\vec{P}(X)(x_0, x_1)$ – since $\text{Rep}_+(I)$ is convex and thus contractible. The map $\vec{N}(X)(x_0, x_1) \rightarrow \vec{T}(X)(x_0, x_1)$ is even a homeomorphism. For details, we refer to Raussen [Rau09b, Section 2.4]. \square

Essentially the same construction was used in the more general setup of Ziemiański [Zie12b].

6.2.2. General properties of trace spaces in a cubical complex. Every connected component of a cubical complex has a topology that is induced from a metric space: A chain connecting two elements $x, y \in X$ in the same component is a sequence $x = x_0, x_1, \dots, x_n = y$ in X such that two subsequent elements x_i, x_{i+1} , $0 \leq i < n$, are contained in the *same* cell. The L_1 -distance $d_1(x, y)$ is then defined as the infimum (in fact the minimum) over all sums $\sum_{i=0}^{n-1} |l_1^\pm(p_i)|$ with p_i any path from x_i to x_{i+1} in a common cell – extending over all chains between x and y . It is easy to check that d_1 is a metric.

The compact-open topology on path space $\vec{P}(X)$ is induced from the supremum metric $d_1(p, q) = \max_{t \in I} d(p(t), q(t))$ for $p, q \in \vec{P}(X)$ – for paths within the same component. By Proposition 6.2.4, trace space $\vec{T}(X)$ is homeomorphic to the subspace $\vec{N}(X)$ and can thus be seen as a subspace of this metric subspace.

PROPOSITION 6.2.5 (Raussen [Rau09b]). *Given a cubical complex X with elements $x_0, x_1 \in X$.*

- (1) *Path space $\vec{P}(X)(x_0, x_1)$ and trace space $\vec{T}(X)(x_0, x_1)$ are metrizable and thus Hausdorff and paracompact.*

(2) If X is a finite complex, then these metric spaces are separable.

A space of d-paths is never compact – unless it only contains constant paths. This is so since the space of reparametrizations $\text{Rep}_+(I) = \vec{P}(I)(0, 1)$ is not compact; it is not even equicontinuous, a necessary condition for compactness by the Arzelà-Ascoli theorem (cf. e.g. [Dug66, Mun75]).

Trace spaces are in general not compact either. If the d-space X contains a non-trivial loop based at $x_0 \in X$, then the closed subspace $\vec{T}(X)(x_0, x_0)$ has d-paths of infinitely many L_1 -arc lengths and thus by Proposition 6.2.2 infinitely many connected components whence it cannot be compact. But compactness results are available if one bounds the L_1 -arc lengths of d-paths:

PROPOSITION 6.2.6 (Raussen, [Rau09b]). *Let X denote a finite cubical complex.*

- (1) *A subset $H \subseteq \vec{T}(X)$ of bounded L_1 -arc length is relatively compact.*
- (2) *For $x_0, x_1 \in X$, every dihomotopy class (connected component) in $\vec{T}(X)(x_0, x_1)$ is compact.*
- (3) *Trace space $\vec{T}(X)$ is locally compact.*

John Milnor investigated in Milnor [Mil59] conditions for spaces that ensure that a variety of mapping spaces have the *homotopy type of a CW-complex*: One may check that these criteria can be applied to spaces of traces in a cubical complex satisfying an extra condition, and we conclude that those spaces have the homotopy type of a CW-complex.

DEFINITION 6.2.7. (Milnor [Mil59]) A topological space A is called ELCX (equi locally convex) provided there are

- (1) a neighborhood U of the diagonal $\Delta A \subset A \times A$ and a map $\lambda : U \times I \rightarrow A$ satisfying
 $\lambda(a, b, 0) = a, \lambda(a, b, 1) = b$ for all $(a, b) \in U$, and
 $\lambda(a, a, t) = a$ for all $a \in A, t \in I$;
- (2) an open covering of A by sets V_β such that $V_\beta \times V_\beta \subset U$ and
 $\lambda(V_\beta \times V_\beta \times I) = V_\beta$.

LEMMA 6.2.8. (a special case of [Mil59, Lemma 4])

Every paracompact ELCX space has the homotopy type of a CW-complex. □

In fact, Milnor shows that a paracompact ELCX space is dominated by a simplicial complex and thus (see e.g. Hatcher [Hat02, Appendix, Proposition A.11]) homotopy equivalent to a CW-complex.

PROPOSITION 6.2.9 (Raussen, [Rau09b]). *Let X denote a non-self linked cubical complex (cf Definition 3.1.3) and let $x_0, x_1 \in X$. Then*

- (1) *X is ELCX.*
- (2) *The path spaces $\vec{P}(X)$ and $\vec{P}(X)(x_0, x_1)$ are ELCX.*
- (3) *The path and trace spaces $\vec{P}(X), \vec{P}(X)(x_0, x_1), \vec{T}(X)$ and $\vec{T}(X)(x_0, x_1)$ have the homotopy type of a CW-complex.*

6.3. Outlook and discussion

It is a natural but unfortunately still unachieved project to try to achieve a generalisation of the results on reparametrizations of paths (and d-paths) to mapping spaces with boxes \square^n resp. directed boxes $\vec{\square}^n$ as their domain. A further generalization would concern mapping spaces with domain a cubical complex.

Later work, described in the subsequent Section 7 yields a precise cell structure – as a simplicial complex – for a complex homotopy equivalent to both $\vec{P}(X)(x_0, x_1)$ and $\vec{T}(X)(x_0, x_1)$; in Section 7.1 for a space X arising from a linear semaphore model as introduced in Section 2; then, more generally in Section 7.3, for d-paths in a general cubical complex X .

Homotopy types of trace spaces

In this section, we report on recent work that allows to identify the homotopy types of trace spaces of convenient d-spaces as *explicitly given* (generalized simplicial) complexes. A resulting combinatorial description of the chain complexes associated to these complexes allows to calculate algebraic topological invariants for the trace space under investigation.

It should be admitted right away, though, that the simplicial complexes and chain complexes in question will often be huge. Apart from a few very easy toy cases, actual calculations need the power of dedicated computer algorithms for homology calculations. Experiments using such software (eg CHomP and CrHom) done by Polish collaborators on top of outputs of the dedicated ALCOOL software of French colleagues have shown limits to the calculations of, say, Betti numbers using this approach – apart from the quintessential β_0 , ie the number of components of trace spaces. For an algorithm finding and exploiting the simplicial complexes mentioned above as well as for implementation issues, consult Section 7.1.4.

7.1. Simplicial models for trace spaces associated to semaphore models

It is easiest to describe the general idea for the spaces associated to semaphore models explained in Chapter 2: The state space is of the form $X = I^n \setminus F$ with a forbidden region $F = \bigcup_{i=1}^l R^i$ consisting of isothetic hyperrectangles $R^i = \prod_{j=1}^n [a_j^i, b_j^i]$. To make notation as easy as possible, we concentrate on a description of trace space $\vec{T}(X)(\mathbf{0}, \mathbf{1})$ between bottom $\mathbf{0}$ and top $\mathbf{1}$ in the hypercube I^n – the generalization to the general case $\vec{T}(X)(\mathbf{c}, \mathbf{d})$ is not very difficult, cf Raussen [Rau12a].

The general strategy may be described as follow:

- (1) Decompose X into finitely many subspaces $X_k \subset X$ such that the sub-trace spaces $\vec{T}(X_k)(\mathbf{0}, \mathbf{1})$ cover the entire trace space $\vec{T}(X)(\mathbf{0}, \mathbf{1})$ and such that all these sub-trace spaces and all intersections of those are either *empty* or *contractible*. The non-empty ones among those intersections form a poset category.
- (2) Apply a variant of the nerve lemma (eg Kozlov [Koz08, Theorem 15.21]) to identify the homotopy type of $\vec{T}(X)(\mathbf{0}, \mathbf{1})$ with the nerve of that poset category – a finite simplicial complex.

In fact, one can do a bit better by replacing the poset category by a related smaller category $\mathcal{C}(X)(\mathbf{0}, \mathbf{1})$ taking into account the product structure induced from the collection of individual hyperrectangles; the associated complex is then a prodsimplicial complex in the terminology of Kozlov [Koz08].

It turns out that the method used to detect deadlocks and unsafe regions described in Section 2 comes in very handy to distinguish those sub-trace spaces that are *empty* from those that are *not*.

7.1.1. Contractible subspaces of trace spaces. For bookkeeping purposes, we consider the set $M_{l,n} = M_{l,n}(\mathbb{Z}/2)$ of all *binary* $l \times n$ -matrices – with 2^{ln} elements – and the subset $M_{l,n}^R \subset M_{l,n}$ consisting of the $(2^n - 1)^l$ matrices with the property that *no row vector is a zero vector*. We regard $M_{l,n}$ and $M_{l,n}^R$ as poset categories with the coordinatewise partial order \leq .

For every matrix $M \in M_{l,n}$, we define a subspace X_M of $X = I^n \setminus F$:

DEFINITION 7.1.1.

$$\begin{aligned} X_M &:= \{ \mathbf{x} \in X \mid m_{ij} = 1 \Rightarrow x_j^i \leq a_j^i \text{ or } \exists k: x_k \geq b_k^i \} \\ &= \{ \mathbf{x} \in X \mid \forall i: (\forall k: x_k < b_k^i \Rightarrow (m_{ij} = 1 \Rightarrow x_j^i \leq a_j^i)) \}. \end{aligned}$$

REMARK 7.1.2. (1) Interpretation: An execution path in X_M has the property:

If $m_{ij} = 1$, then process j will *not* acquire a lock to resource i before at least one of the others has relinquished it.

(2) Matrices in $M_{l,n}^R$ represent areas in which *each* hole is obstructed furthermore in *at least one* direction.

EXAMPLE 7.1.3. (1) Figure 7.1 shows, in each of the two rows, an example of a model space $X = \vec{I}^2 \setminus F$ given as the complement of the forbidden region F consisting of two black squares. The grey-shaded areas show, in both cases, the four subspaces X_M corresponding to the four matrices

$$M = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$$

– in that order.

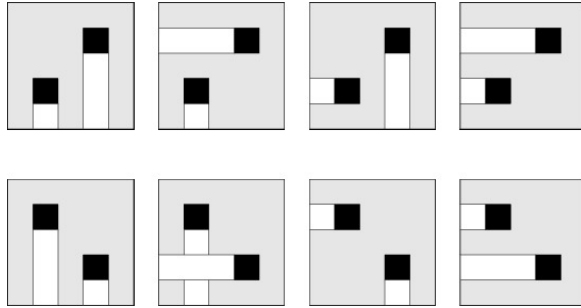


FIGURE 7.1. Two examples of a model space X and of associated subspaces X_M – the grey-shaded areas.

Remark that an empty space of d-paths $\vec{P}(X_M)(\mathbf{0}, \mathbf{1}) = \emptyset$ occurs only in the second row and second column in Figure 7.1. All other spaces $\vec{P}(X_M)(\mathbf{0}, \mathbf{1})$ are non-empty and contractible. This can be used to explain that the trace space $\vec{T}(X)(\mathbf{0}, \mathbf{1})$ has four (contractible) components in the first case and three components in the second.

- (2) Figure 7.2 below shows a forbidden region “black box” \vec{J}^3 – with $\vec{J} \subset \vec{I}$ an interior open interval – with upper corner \mathbf{b} surrounded by the state space $X = \vec{I}^3 \setminus \vec{J}^3$. Moreover, you recognize the shaded areas $X_{M_j} \cap (\partial_+ \downarrow \mathbf{b})$, $1 \leq j \leq 3$, with $M_1 = [100]$, $M_2 = [010]$ and $M_3 = [001]$.

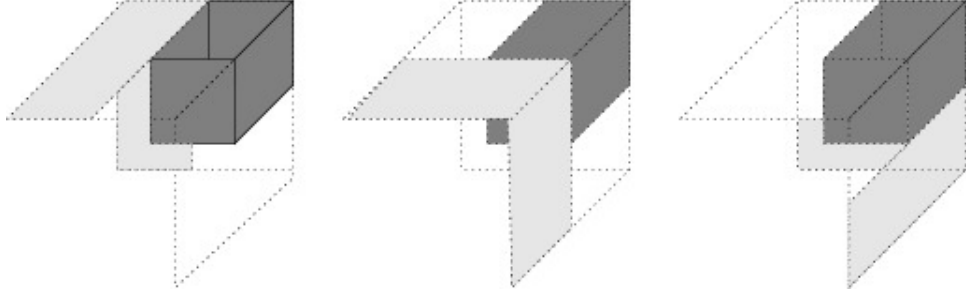


FIGURE 7.2. Intersections of X_{M_j} with the upper boundary $\partial_+ \downarrow \mathbf{b}$ of the black box $\downarrow \mathbf{b}$ with upper corner \mathbf{b} .

Remark that every pair of these areas intersect non-trivially, whereas the intersection $X_{[111]} = X_{M_1} \cap X_{M_2} \cap X_{M_3}$ is empty. In particular, $\vec{P}(X_M)(\mathbf{0}, \mathbf{1}) = \emptyset$ for $M \in M_{1,3}^R$ if and only if $M = [111]$. The subsequent analysis yields as a consequence that $\vec{T}(X)(\mathbf{0}, \mathbf{1}) \simeq \partial\Delta^2 \cong S^1$.

The following constructions depend on an analysis of the binary operation \vee (least upper bound) defined on \mathbf{R}^n : $\mathbf{a} \vee \mathbf{b} = (\max(a_1, b_1), \dots, \max(a_n, b_n))$. This operation restricts to an operation on I^n , but its restriction to $X \times X$ has values in the forbidden region F . This is why we subdivide X into subspaces X_M , $M \in M_{l,n}^R$:

- PROPOSITION 7.1.4. (1) The subspaces X_M , $M \in M_{l,n}^R$ are all closed under \vee .
 (2) $\vec{T}(X)(\mathbf{0}, \mathbf{1}) = \bigcup_{M \in M_{l,n}^R} \vec{T}(X_M)(\mathbf{0}, \mathbf{1})$ – every trace is contained in at least one of the restricted regions X_M .
 (3) Every trace space $\vec{T}(X_M)(\mathbf{0}, \mathbf{1})$, $M \in M_{l,n}^R$, is empty or contractible.

PROOF. For details, we refer to Raussen [Rau10].

- (1) Verify that the inequalities defining X_M are satisfied for $\mathbf{a} \vee \mathbf{b}$, $\mathbf{a}, \mathbf{b} \in X_M$. It is crucial that $M \in M_{l,n}^R$ – the inequalities “guard” every hole R^i .
- (2) For a given d-path $p \in \vec{P}(X)(\mathbf{0}, \mathbf{1})$, let $t_i^+ = \min\{t \mid \exists k : p_k(t) = b_k^i\}$ for $1 \leq i \leq l$. Then there exists $j \in [1 : n]$ such that $p_j(t_i^+) \leq a_j^i$ and hence $p_j(t) \leq a_j^i$ for $t \leq t_i^+$; otherwise $p(t) \in R^i$ on a non-empty interval $]t_i^-, t_i^+[$.
- (3) If $\vec{T}(X_M)(\mathbf{0}, \mathbf{1})$ is non-empty, then, for any pair $p, q \in \vec{P}(X_M)(\mathbf{0}, \mathbf{1})$, define a one-parameter family $H(p, q) : \vec{P}(X_M)(\mathbf{0}, \mathbf{1}) \times I \rightarrow \vec{P}(X_M)(\mathbf{0}, \mathbf{1})$ by

$$H_t(p, q)(s) := q(s) \vee p(ts), \quad t \in I.$$

Remark that $H_0(p, q)(s) = q(s) \vee \mathbf{0} = q(s)$, $H_t(p, q)(0) = \mathbf{0} \vee \mathbf{0} = \mathbf{0}$, $H_t(p, q)(1) = \mathbf{1} * p(t) = \mathbf{1}$ and that $H_1(p, q)(s) = q(s) \vee p(s)$. Thus $H(p, q)$ defines an increasing d-homotopy (cf Grandis [Gra03a]) $q \mapsto p \vee q$ between d-paths within $\vec{P}(X_M)(\mathbf{0}, \mathbf{1})$. Likewise, $H(q, p)$ is an increasing d-homotopy $p \mapsto q \vee p = p \vee q$. Their concatenation $G(q, p) = H(p, q) * H^-(q, p)$ (orientations are reversed for the second d-homotopy) is a “zig-zag” d-homotopy from q to p ; in particular a path from q to p within $\vec{P}(X_M)(\mathbf{0}, \mathbf{1})$.

The map $G(-, -) : \vec{P}(X_M)(\mathbf{0}, \mathbf{1}) \times \vec{P}(X_M)(\mathbf{0}, \mathbf{1}) \rightarrow \vec{P}(X_M)(\mathbf{0}, \mathbf{1})^I$ defines a continuous section of the “end path map”

$$ev_0 \times ev_1 : \vec{P}(X_M)(\mathbf{0}, \mathbf{1})^I \rightarrow \vec{P}(X_M)(\mathbf{0}, \mathbf{1}) \times \vec{P}(X_M)(\mathbf{0}, \mathbf{1});$$

it associates the d-homotopy $G(p, q)$ to a pair (q, p) .

Given an arbitrary $p \in \vec{P}(X_M)(\mathbf{0}, \mathbf{1})$, the map $G(-, p) : \vec{P}(X_M)(\mathbf{0}, \mathbf{1}) \times I \rightarrow \vec{P}(X_M)(\mathbf{0}, \mathbf{1})$ is a contraction of $\vec{P}(X_M)(\mathbf{0}, \mathbf{1})$ to p . By Raussen [Rau09b, Proposition 2.16] or Proposition 6.2.4(2), the trace space $\vec{T}(X_M)(\mathbf{0}, \mathbf{1})$ is homotopy equivalent to path space $\vec{P}(X_M)(\mathbf{0}, \mathbf{1})$; hence it is also contractible. \square

7.1.2. A variant of the nerve lemma leads to a prodsimplicial complex. In the following, we will work with a restriction of the poset category $M_{l,n}$ of binary matrices from Section 7.1.1: The relevant index category to consider here is the full subposet category $\mathcal{C}(X)(\mathbf{0}, \mathbf{1}) \subset M_{l,n}^R \subset M_{l,n}$ consisting of all matrices M such that

$$(7.1) \quad \vec{T}(X_M)(\mathbf{0}, \mathbf{1}) \neq \emptyset.$$

This index category gives rise to functors \mathcal{D} and \mathcal{E} into **Top**:

- DEFINITION 7.1.5.** • For a non-zero binary vector $\mathbf{m} \in (\mathbf{Z}/2)^n$, let $\Delta(\mathbf{m}) \subseteq \Delta^{n-1}$ denote the simplex spanned by the unit vectors $\mathbf{e}_j \in \mathbf{R}^n$ with $m_j = 1$.
- For $M \in M_{l,n}^R$, let $\Delta(M) = \prod_{i=1}^l \Delta(\mathbf{m}_i) \subseteq (\Delta^{n-1})^l$.
 - The functor $\mathcal{D} : \mathcal{C}(X)(\mathbf{0}, \mathbf{1})^{op} \rightarrow \mathbf{Top}$ associates $\vec{T}(X_M)(\mathbf{0}, \mathbf{1})$ to the matrix M ; the reverse partial order on $\mathcal{C}(X)(\mathbf{0}, \mathbf{1})$ corresponds to inclusion in **Top**.
 - The functor $\mathcal{E} : \mathcal{C}(X)(\mathbf{0}, \mathbf{1}) \rightarrow \mathbf{Top}$ restricts from a functor $\mathcal{E}_n^l : M_{l,n}^R \rightarrow \mathbf{Top}$; it associates to $M \in \mathcal{C}(X)(\mathbf{0}, \mathbf{1}) \subseteq M_{l,n}^R$ the simplex product $\Delta(M)$. For this functor, the *original* partial order on $\mathcal{C}(X)(\mathbf{0}, \mathbf{1})$ corresponds to inclusion in **Top**.

The functor \mathcal{E}_n^l should be considered as a pasting scheme for the product of simplices $(\Delta^{n-1})^l$; the functor \mathcal{E} becomes then a pasting scheme for a sub-prodsimplicial complex (cf Kozlov [Koz08]) $\mathbf{T}(X)(\mathbf{0}, \mathbf{1}) \subseteq (\Delta^{n-1})^l$ to be explained below.

Regarding the functors \mathcal{E} and \mathcal{D} as pasting schemes, we consider their colimits:

- $\text{colim}(\mathcal{D}) = \vec{T}(X)(\mathbf{0}, \mathbf{1})$ – by Proposition 7.1.4;
- $\text{colim}(\mathcal{E}_n^l) = (\Delta^{n-1})^l$;

- $\mathbf{T}(X)(\mathbf{0}, \mathbf{1}) := \operatorname{colim}(\mathcal{E}) \subset \operatorname{colim}(\mathcal{E}_n^l) = (\Delta^{n-1})^l$ is a *prodsimplicial* complex consisting of all those products of simplices $\Delta(M)$ that correspond to matrices $M \in \mathcal{C}(X)(\mathbf{0}, \mathbf{1})$; in other words, the functor \mathcal{E} is a pasting scheme for a prodsimplicial complex with one simplex product for each $M \in M_{l,n}^R$ giving rise to a *non-empty* trace space $\vec{T}(X_M)(\mathbf{0}, \mathbf{1})$.
- As a subcomplex of $(\partial\Delta^{n-1})^l \cong (S^{n-2})^l$, the prodsimplicial complex $\mathbf{T}(X)(\mathbf{0}, \mathbf{1})$ has at most n^l vertices, and $\dim(\mathbf{T}(X)(\mathbf{0}, \mathbf{1})) \leq (n-2)l$.

EXAMPLE 7.1.6. Figure 7.3 shows the state space X from Section 3.2.4 with a (slightly compressed) picture of $\mathbf{T}(X)(\mathbf{0}, \mathbf{1}) \subset \partial\Delta^2 \times \partial\Delta^2$. It consists of the shaded parts of the torus and the isolated vertex shown four times in the covering (corresponding to the contractible component of the path space with the d-path in the illustration on the left hand side). That latter model space is clearly homotopy equivalent to $S^1 \vee S^1 \sqcup *$.

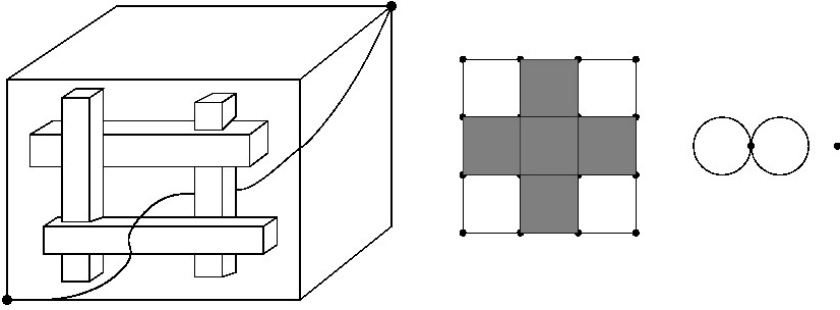


FIGURE 7.3. State space X and prodsimplicial model $\mathbf{T}(X)(\mathbf{0}, \mathbf{1})$ of the path space

THEOREM 7.1.7. *The trace space $\vec{T}(X)(\mathbf{0}, \mathbf{1})$ is homotopy equivalent to the prodsimplicial complex $\mathbf{T}(X)(\mathbf{0}, \mathbf{1}) \subset (\partial\Delta^{n-1})^l$ and to the nerve of the category $\mathcal{C}(X)(\mathbf{0}, \mathbf{1})$; the latter simplicial complex arises as a barycentric subdivision of $\mathbf{T}(X)(\mathbf{0}, \mathbf{1})$.*

PROOF. First, we determine the *homotopy* colimits of the functors defining the pasting schemes above. We apply the homotopy lemma (cf eg Kozlov [Koz08, Theorem 15.12]) to the natural transformation $\Psi : \mathcal{D} \Rightarrow \mathcal{T}^*$ from \mathcal{D} to the trivial functor $\mathcal{T}^* : \mathcal{C}(X)(\mathbf{0}, \mathbf{1})^{op} \rightarrow \mathbf{Top}$ which sends every object into the same one-point space. Since the maps corresponding to Ψ are homotopy equivalences at every object M in $\mathcal{C}(X)(\mathbf{0}, \mathbf{1})$ (from a contractible space $\vec{T}(X_M)(\mathbf{0}, \mathbf{1})$ – by Proposition 7.1.4(3) – to a point), the map $\operatorname{hocolim} \mathcal{D} \rightarrow \operatorname{hocolim} \mathcal{T}^*$ induced by Ψ is a homotopy equivalence by the homotopy lemma. By definition, $\operatorname{hocolim} \mathcal{T}^*$ is the nerve $\Delta(\mathcal{C}(X)(\mathbf{0}, \mathbf{1}))$ of the indexing category.

A similar argument shows that also the trivial natural transformation from \mathcal{E} to $\mathcal{T} : \mathcal{C}(X)(\mathbf{0}, \mathbf{1}) \rightarrow \mathbf{Top}$ induces a homotopy equivalence of homotopy colimits.

Next, we wish to apply the projection lemma (cf Segal [Seg68, Proposition 4.1] or Kozlov [Koz08, Theorem 15.19]) to the fiber projection maps $\operatorname{hocolim} \mathcal{D} \rightarrow \operatorname{colim} \mathcal{D}$ and $\operatorname{hocolim} \mathcal{E} \rightarrow \operatorname{colim} \mathcal{E}$. The given cover is not open, but it was shown in Raussen [Rau10] how it can be replaced by an open cover of homotopy equivalent spaces. Hence the projection lemma ensures that these maps are homotopy

equivalences. Altogether, the maps discussed above fit to yield a homotopy equivalence

$$\begin{array}{ccc} \vec{T}(X)(\mathbf{0}, \mathbf{1}) = \operatorname{colim}(\mathcal{D}) & \longleftarrow \operatorname{hocolim}(\mathcal{D}) \longrightarrow & \operatorname{hocolim}(\mathcal{T}^*) \\ & & \updownarrow \\ \mathbf{T}(X)(\mathbf{0}, \mathbf{1}) = \operatorname{colim}(\mathcal{E}) & \longleftarrow \operatorname{hocolim}(\mathcal{E}) \longrightarrow & \operatorname{hocolim}(\mathcal{T}) \end{array}$$

since the two opposite categories $\mathcal{C}(X)(\mathbf{0}, \mathbf{1})$ and $\mathcal{C}(X)(\mathbf{0}, \mathbf{1})^{op}$ have the same classifying space $\Delta(\mathcal{C}(X)(\mathbf{0}, \mathbf{1}))$. In particular, $\mathbf{T}(X)(\mathbf{0}, \mathbf{1})$ is also homotopy equivalent to the nerve $\Delta(\mathcal{C}(X)(\mathbf{0}, \mathbf{1}))$ – which is thus a barycentric subdivision of $\mathbf{T}(X)(\mathbf{0}, \mathbf{1})$. \square

7.1.3. Determination of dead and of alive matrices. In the following, we call a matrix $M \in M_{l,n}^R$ *alive* if $\vec{T}(X_M)(\mathbf{0}, \mathbf{1}) \neq \emptyset$ and *dead* else. It is crucial for an algorithmic description of $\mathcal{C}(X)(\mathbf{0}, \mathbf{1})$ to find a method that distinguishes dead and alive matrices. It turns out that the determination of dead matrices may be achieved through the method determining deadlocks and unsafe regions from Chapter 2. But first we need to establish an easy order property:

Consider the map $\Psi : M_{l,n} \rightarrow \mathbf{Z}/2$, $\psi(M) = 1 \Leftrightarrow \vec{T}(X_M)(\mathbf{0}, \mathbf{1}) = \emptyset$ and the subset $M_{l,n}^C \subset M_{l,n}$ consisting of all matrices with *unit vectors* as columns – minimal *candidates* for dead matrices. Then

- PROPOSITION 7.1.8. (1) Ψ is order-preserving.
 (2) $\Psi(M) = 1 \Leftrightarrow$ there exists $N \in M_{l,n}^C$ with $\Psi(N) = 1$ and $N \leq M$.

For the easy proof, we refer to Raussen [Rau10, Proposition 4.5].

The first property in Proposition 7.1.8 tells us that $\mathcal{C}(X)(\mathbf{0}, \mathbf{1})$ is indeed closed under containment and its geometric realization is thus a complex.

By the second property, we can concentrate on determining the subset $D(X)(\mathbf{0}, \mathbf{1}) := \{M \in M_{l,n}^C \mid \Psi(M) = 1\} \subset M_{l,n}^C$. It allows to describe $\mathcal{C}(X)(\mathbf{0}, \mathbf{1})$ as the set of matrices $M \in M_{l,n}^R$ with the property: For every matrix $N \in D(X)(\mathbf{0}, \mathbf{1})$, there is a pair $(i, j) \in [1 : l] \times [1 : n]$ such that $m_{ij} = 0, n_{ij} = 1$.

Since the map Ψ above is order-preserving, it is enough to describe the *maximal* matrices in $\mathcal{C}_{\max}(X)(\mathbf{0}, \mathbf{1}) \subseteq \mathcal{C}(X)(\mathbf{0}, \mathbf{1})$ that are “just alive”: replacing just one entry $m_{ij} = 0$ by an entry 1 makes such a matrix greater or equal than a dead matrix.

The crucial idea for the determination of the *dead* matrices in $D(X)(\mathbf{0}, \mathbf{1}) \subset M_{l,n}^C$ is a variant of the algorithm determining deadlocks and unsafe regions from Chapter 2: The aim is to describe the subspaces $X_M \subseteq X$ as *complements* of a union of *extended* hyperrectangles of type

$$R_j^i = \prod_{k=1}^{j-1} \tilde{I}_k^i \times I_j^i \times \prod_{k=j+1}^n \tilde{I}_k^i, \quad 1 \leq i \leq l, 1 \leq j \leq n$$

with $\tilde{I}_k^i = [0, b_k^i[\square]a_k^i, b_k^i[$. It is then easy to see (Raussen, [Rau10, Lemma 4.2]) that $X_M = \tilde{I}^n \setminus \bigcup_{m_{ij}=1} R_j^i$.

Furthermore, as soon as n extended hyperrectangles R_j^i , one for each $1 \leq j \leq n$, give rise to a deadlock, the associated unsafe region is a hyperrectangle with $\mathbf{0}$

as the lowest vertex: the second largest coordinates (cf Section 2.2.1) are all 0 in extended hyperrectangles. In the end, everything boils down to a systematic check of various sets of inequalities between bottom and top coordinates a_j^i and $b_j^{i'}$ of intervals in the product decomposition of the original hyperrectangles, cf Raussen [Rau10, Section 4].

7.1.4. Implementation issues. The algorithm described above that determines the poset category $\mathcal{C}(X)(\mathbf{0}, \mathbf{1})$ has been implemented in the ALCOOL tool of our French partners at CEA Saclay, cf Fajstrup et al [FGH⁺12]. That latter paper contains also considerations about how to extend the methods from this section to cubical complexes that arise from the spaces X_M by identifying boundary faces, ie, complexes that are subspaces of products of a torus and a box arising by deletion of forbidden hyperrectangles.

The ALCOOL tool produces (among other deliveries) a description of the maximal alive matrices in $\mathcal{C}_{\max}(X)(\mathbf{0}, \mathbf{1})$. The boundary operator ∂ of an associated chain complex (with $\mathbf{Z}/2$ coefficients) can be implemented by a sum of terms in which exactly one of the digits 1 is replaced by a zero. In this way, it was possible for M. Juda (Krakow) to adapt the homology software created by the Polish group around M. Mrozek (cf eg Kaczynski et al [KMM04]) to do homology calculations of trace spaces.

This works well for semaphore protocols of a very moderate size. Unfortunately, when the number l of obstructions grows, the model $\mathbf{T}(X)(\mathbf{0}, \mathbf{1})$ becomes quickly high-dimensional – although the homological dimension of the trace space is conjectured to be far more limited. The method thus does not yield algorithmically satisfactory results for large l . We work currently on modifications of the method – that needs new theoretical insights – with the aim to reduce the dimension of the complex and thus to make homology calculations feasible in cases of interest.

7.2. Specific results for mutex semaphores – arity one

In the previous Section 7.1, we assumed for simplicity that none of the forbidden hyperrectangles R^i intersects the boundary ∂I^n . For semaphore models, this is true only for semaphores of arity $n - 1$, giving simultaneous access to $n - 1$ but not to n of the processors. For semaphores of a lower arity a (cf Section 2.1), one shared object gives rise to a union of $\binom{n}{a+1}$ hyperrectangles each of which contains $n - a - 1$ maximal interval factors $[0, 1]$ – and thus intersections with the boundary.

This fact, and also a need for investigation of “intermediate” trace spaces $\vec{T}(X)(\mathbf{c}, \mathbf{d})$, $\mathbf{0} \leq \mathbf{c} \leq \mathbf{d} \leq \mathbf{1}$, motivated the extension of the method described in Section 7.1 to more general forbidden regions developed in Raussen [Rau12a]. We will not try to describe the quite technical development needed in that contribution in general; instead we list some results in the particularly interesting case where all semaphores have arity *one*: Only one process can access a shared object at any given time. This is the particularly important case of *mutual exclusion* or *mutex* semaphores. In this case, we prove:

PROPOSITION 7.2.1. (Raussen [Rau12a, Proposition 7]) *Let $X = I^n \setminus F$ denote the state space corresponding to a collection C of calls to semaphores of arity one. Then the trace space $\vec{T}(X)(\mathbf{0}, \mathbf{1})$ is homotopy equivalent to a finite discrete space.*

We knew this already for $n = 2$ – in this case semaphores can only have arity one: The prodsimplicial complexes considered in Section 7.1 are then all subcomplexes of a product of 0-dimensional spheres.

For the state space X corresponding to a program with just a *single call* to one semaphore of arity one, this can be phrased more specifically as follows: X may be decomposed into subspaces X_π , one for every permutation $\pi \in \Sigma_n$. More specifically, we show for this case:

COROLLARY 7.2.2. (Raussen [Rau12a, Corollary 2]) *The trace space is a disjoint union $\vec{T}(X)(\mathbf{0}, \mathbf{1}) = \bigsqcup_{\pi \in \Sigma_n} \vec{T}(X_\pi)(\mathbf{0}, \mathbf{1})$. All $n!$ components $\vec{T}(X_\pi)(\mathbf{0}, \mathbf{1})$, $\pi \in \Sigma_n$, are contractible.*

The general situation (more than one call) is far more complex, but it can in the end be translated to the discrete realm using a notion of *compatible permutations*: Every semaphore h can be called several times by a number of processes; each concurrent call c is performed by a subset $J_h \subset [1 : n]$ of (at least two competing) processes; a call c is characterized by a semaphore $h(c)$, by the subset $J_{h(c)}$ and by one out of $r_j(h(c))$ locking intervals on the axes corresponding to $J_{h(c)}$. Every such call $c = (h; m_1(h), \dots, m_{|J_h|}(h))$, $1 \leq m_j(h) \leq r_j(h)$, gives rise to a forbidden region $F(c)$. Such a forbidden region alone would give rise to a trace space homotopy equivalent to a discrete space $\Sigma_{J_{h(c)}} \subset \Sigma_n$ – the stabilizer of $[1 : n] \setminus \Sigma_{J_{h(c)}}$, with cardinality $|J_{h(c)}|!$.

In total, we have to study the complement of a forbidden region $F = \bigcup_{c \in C} F(c)$ with C denoting the set of *all* calls. This suggests to study *collections* of permutations $\pi = (\pi_c)_{c \in C} \in \Sigma = \prod_{c \in C} \Sigma_{J_{h(c)}}$: Consider the set of boundary coordinates $a_j^i, b_j^i \in I$, $1 \leq j \leq n$, corresponding to *all* concurrent calls to the semaphores. For every collection $\pi = (\pi_c)_{c \in C} \in \Sigma = \prod_{c \in C} \Sigma_{J_{h(c)}}$, we consider several order relations on subsets of these real numbers:

- The natural order \leq , inherited from the reals, on numbers a_j^i, b_j^i with the *same* subscript (direction) j ;
- $b_{\pi_c(j)}^{m_{\pi_c(j)}(h)} \preceq a_{\pi_c(j')}^{m_{\pi_c(j')}(h)}$ for $c \in C$, $j < j' \in J_{h(c)}$ for the *same* call $c = (h; m_1(h), \dots, m_{|J_h|}(h)) \in C$.

DEFINITION 7.2.3. We call the collection $\pi = (\pi_c)_{c \in C} \in \Sigma = \prod_{c \in C} \Sigma_{J_{h(c)}}$ *compatible* if the transitive closure \sqsubseteq_π of these relations is a *partial order*.

This notion was applied in the proof of

PROPOSITION 7.2.4. (Raussen, [Rau12a, Proposition 10]) *Let $X = I^n \setminus F$ denote the state space corresponding to a collection C of calls to semaphores of arity one.*

Then $\vec{T}(X)(\mathbf{0}, \mathbf{1})$ is homotopy equivalent to the discrete space

$$\{\pi = (\pi_c)_{c \in C} \in \prod_{c \in C} \Sigma_{J_{h(c)}} \mid \pi \text{ compatible}\} \subseteq \prod_{c \in C} \Sigma_{J_{h(c)}} \subseteq (\Sigma_n)^{|C|}.$$

EXAMPLE 7.2.5. Let $X_k \subset I^k$ denote the state space corresponding to the PV-model describing k dining philosophers (each protocol of type $PaPbVaVb$; cf Dijkstra [Dij71] and Section 2.1.1. Then the trace space $\vec{T}(X_k)(\mathbf{0}, \mathbf{1})$ consists of $2^k - 2$ contractible components: There are $2^k - 2$ essentially different interleavings of the d-paths corresponding to each individual protocol – indicating who of two

neighbouring philosophers uses a fork first. The number $2^k - 2$ of schedules is, for $k > 3$, considerably smaller than the number $k!$ of ordered k -tuples of philosophers. This is due to the fact that several philosophers can serve themselves concurrently for $k > 3$. For a detailed analysis, cf Raussen [Rau12a, Example 2].

7.3. General Higher Dimensional Automata

In the preceeding sections, methods have only been worked out for semaphore models *without* loops; an important, but restricted family of general Higher Dimensional Automata (HDA). It turned out that some general ideas from this analysis can be applied to the general case, as well. In particular, the general case includes HDA with non-trivial directed *loops*, absolutely essential in the analysis of realistic concurrent programs. It has to be admitted, that the ideas that we describe below are certainly far more difficult to implement; this has not even been tried so far.

The aim is to find a combinatorial model $\mathbf{T}(X)(\mathbf{x}_0, \mathbf{x}_1)$ of the trace space $\vec{T}(X)(\mathbf{x}_0, \mathbf{x}_1)$ for a general non-self-linked (cf Fajstrup et al [FGR06], Raussen [Rau09b]) cubical complex X . This is done in several steps:

7.3.1. Trace spaces for non-looping cubical complexes. First we study trace spaces for non-self-linked cubical complexes *without* (non-trivial) d-loops and look for a replacement of the subspaces X_M from Section 7.1. These turn out to be (maximal) *non-branching* (sub)-complexes:

A (finite) cubical complex (geometric realization of a pre-cubical set) X will be called *non-branching* if it satisfies the following additional property

(NB): Every vertex $v \in X_0$ is the lower corner vertex of a *unique maximal* cube c_v in X . This maximal cube c_v contains thus all cubes with lower corner vertex v as a (possibly iterated) lower face.

PROPOSITION 7.3.1. *For every pair of elements x_0, x_1 in a non-branching cubical complex X , the trace space $\vec{T}(X)(x_0, x_1)$ is either empty or contractible.*

Instead of using the least upper bound (\vee) operation (essential in the proof of Proposition 7.1.4), we apply the diagonal *directed flow* $F_c^X: X \times \mathbf{R}_{\geq 0} \rightarrow X$: Every element $x \in X$ is contained in the interior or the lower boundary of a uniquely determined maximal cube, i.e., the maximal cube c_v of its lowest vertex v . On the interior and the lower faces of such a cube c_v , this flow is locally given by the *diagonal flow*:

$$(7.2) \quad F_c^X(c; (x_1, \dots, x_n); t) = (c; x_1 + t, \dots, x_n + t) \text{ for } 0 \leq t \leq 1 - \max_{1 \leq i \leq n} x_i.$$

On a maximal vertex v_1 with $c = c_{v_1} = v_1$ (a deadlock), F_c^X is defined to be constant in the variable t for $0 \leq t$.

Note that property (NB) is essential: necessary and sufficient for pasting diagonal flows together from flows on individual cubes. Diagonal flows on intersecting different *maximal* cubes do not fit together on intersections of their lower boundaries. For a different description using a diagonal 1-form ω as in Section 6.2.1, we refer to Raussen [Rau12b].

One may now construct a cover of a general cubical complex X by maximal subspaces satisfying property (NB) and identify $\vec{T}(X)(\mathbf{x}_0, \mathbf{x}_1)$ with the nerve of that covering; cf Raussen [Rau12b, Theorem 4.2]. To determine such maximal (NB)

subspaces algorithmically, one investigates the branch points in the 0-skeleton X_0 of the complex X – with several maximal cubes having that branch point as lowest vertex – and associated branches, one for every such maximal cube.

Simple examples show that these smaller branch subspaces may have (secondary) branch points. Hence, one has to iterate the construction. One ends up with a poset category $\mathcal{C}(X)(\mathbf{x}_0, \mathbf{x}_1)$ the objects of which are given by so-called coherent and complete sequences of first and higher order branch points and associated choices of branch cubes. This category can be realized as a colimit $\mathbf{T}(\mathbf{x}_0, \mathbf{x}_1)$ of spaces each of which is a product of products of simplices and of cones on such products. In analogy with Theorem 7.1.7, one obtains

THEOREM 7.3.2. *For a cubical complex X without non-trivial loops and points $\mathbf{x}_0, \mathbf{x}_1 \in X$, the trace space $\bar{T}(X)(\mathbf{x}_0, \mathbf{x}_1)$ is homotopy equivalent to*

- (1) *the nerve $\Delta(\mathcal{C}(X)(\mathbf{x}_0, \mathbf{x}_1))$ of the poset category $\mathcal{C}(X)(\mathbf{x}_0, \mathbf{x}_1)$, and*
- (2) *the complex $\mathbf{T}(X)(\mathbf{x}_0, \mathbf{x}_1)$.*

For the proof, we refer to Raussen, [Rau12b, Theorem 4.11].

7.3.2. Trace spaces for cubical complexes with directed loops. We outline how previous methods can be adapted to trace spaces in a general cubical complex X with directed loops using suitable *coverings* of the complex X :

We exploit the d-map $s : X \rightarrow \bar{S}^1 \cong \mathbf{R}/\mathbf{Z}$ introduced in Raussen [Rau09b] and described here in Section 6.2.1: just glue the maps $s(x_1, \dots, x_n) = \sum x_i \bmod 1$ on individual cubes. Consider the pullback \tilde{X} in the pullback diagram

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{s} & X \times \mathbf{R} \\ \pi \downarrow & & \downarrow id \times exp \\ X & \xrightarrow{id \times s} & X \times S^1 \end{array}.$$

The map π is a covering map with unique path lifting. Since exp can be interpreted as a semi-cubical map, \tilde{X} can be conceived as a cubical complex: Every cube e in X is replaced by infinitely many cubes $(e, n), n \in \mathbf{Z}$; the boundary maps are given by $\partial_-(e, n) = (\partial_- e, n), \partial_+(e, n) = (\partial_+ e, n + 1)$.

The directed paths on \tilde{X} are those that project to directed paths in X under the projection map π . Remark that the maps exp and s – and hence π and $\pi_2 \circ S$ – preserve the signed L_1 -arc length l_1^\pm from Section 6.2.1. Moreover, the L_1 -length $l_1^\pm(p)$ of a path p in X with lift \tilde{p} can be expressed via the d-map $S : \tilde{X} \rightarrow X \times \mathbf{R}$ in the pullback diagram as follows:

- LEMMA 7.3.3.**
- (1) $l_1^\pm(p) = \pi_2(S(\tilde{p}(1))) - \pi_2(S(\tilde{p}(0)))$.
 - (2) *The map $\pi_2 : \tilde{X} \rightarrow \mathbf{R}$ is a d-map; hence:*
 - (3) *\tilde{X} has only trivial directed loops.*

Another method to construct this covering is to consider the *homotopical* length map $\pi_1(X) \xrightarrow{l_1^\pm} \mathbf{Z} \rightarrow 0$ (cf Proposition 6.2.2) from the non-directed classical fundamental group of the cubical complex X . Consider the cover $\tilde{X} \downarrow X$ with fundamental group $\pi_1(\tilde{X}) = K \trianglelefteq \pi_1(X)$ the kernel of the homotopical length map l_1^\pm . It can be given the structure of a cubical complex, and every element \mathbf{x} in X is covered

by elements $\mathbf{x}^n \in \tilde{X}$; one for every $n \in \mathbf{Z}$. The projection map $\pi : \tilde{X} \downarrow X$ preserves the signed L_1 -arc length. A path in \tilde{X} is directed if and only if its projection to X is directed. There are no non-trivial directed loops in \tilde{X} – these need to have L_1 -length 0!

For a general cubical complex X with length cover \tilde{X} we obtain the following decomposition result:

PROPOSITION 7.3.4. *For every pair of points $\mathbf{x}_0, \mathbf{x}_1 \in X$, trace space $\vec{T}(X)(\mathbf{x}_0, \mathbf{x}_1)$ is homeomorphic to the disjoint union $\bigsqcup_{n \in \mathbf{Z}} \vec{T}(\tilde{X})(\mathbf{x}_0^0, \mathbf{x}_1^n)$.*

Since the covering \tilde{X} has *only trivial loops*, Proposition 7.3.4 allows us to apply the methods from Section 7.3.1 to describe the homotopy type of trace spaces $\vec{T}(X)(\mathbf{x}_0, \mathbf{x}_1)$ in an arbitrary cubical complex X .

In practice, we have so far investigated simple semaphore models with loops of the form $X = T^n \setminus F$ with $T^n = (S^1)^n$ an n -torus and F a collection of forbidden hyperrectangles. For such a space, one may consider the covering

$$(7.3) \quad \begin{array}{ccc} \tilde{X} & \hookrightarrow & \mathbf{R}^n \\ \downarrow & & \downarrow \text{exp} \\ X & \hookrightarrow & T^n \end{array}$$

that arises as pullback from the universal cover of the torus T^n – a far bigger gadget. The universality property ensures that (d-)paths, that are not homotopic in the torus T^n , lift to (d-)paths with different end points. The methods from Raussen [Rau10] can be applied to \tilde{X} immediately. It is easier to get hold on periodicity properties in this setting; cf Fajstrup et al [FGH⁺12], Raussen and Ziemiański [RZ14] and the following Section 7.4.

7.4. Explicit homology calculations for specific path spaces – with loops

It seems to be difficult to implement the considerations from Section 7.3 above in a working programme for spaces of directed paths with non-trivial directed loops. This is why we have attempted to investigate simple examples in order to find clues for calculations. One of these examples concerns path spaces in a Euclidean torus $T^n = (\vec{S}^1)^n$ from which just one rectangular hole I^n has been removed: $X = T^n \setminus I^n$.

The covering space arising from the construction (7.3) has the description $\tilde{X} = \mathbf{R}^n \setminus \bigsqcup_{\mathbf{c} - \frac{1}{2}\mathbf{1} \in \mathbf{Z}^n} I_{\mathbf{c}}$ with $I_{\mathbf{c}}$ a homothetic hyperrectangle centered at \mathbf{c} – and edges of length less than 1. The space \tilde{X} is homotopy equivalent to the $(n - 1)$ -skeleton of \mathbf{R}^n seen as a cubical complex – with vertices in the integral points. Moreover, inclusion and retraction establishing such a homotopy equivalence can be chosen to preserve directed paths.

Also in this case it is easy to see, cf Raussen and Ziemiański [RZ14, Section 1.5] that the space of directed loops $\vec{P}(X)(\mathbf{x}_0, \mathbf{x}_0)$ based at $\mathbf{x}_0 \in X$ is homotopy equivalent to the disjoint union of spaces $\vec{P}(\tilde{X})(\mathbf{0}, \mathbf{k})$, $\mathbf{k} \geq \mathbf{0}, \mathbf{k} \in \mathbf{Z}^n$. For $n > 2$, this decomposition is also a decomposition into path components; for $n = 2$, $\vec{P}(X)$ is homotopy discrete cf Raussen and Ziemiański [RZ14, Section 1.7] and Corollary 7.4.2.

For $n = 3$, an attempt to calculate the homology of $\vec{P}(\tilde{X})_0^{(k,l,m)}$ by “brute force” using the poset description for the cell complex of the prod-simplicial complex homotopy equivalent to that path space as in Section 7.1 – even using sophisticated homology software – failed already for $k = l = m = 3$. The prod-simplicial complex in this case has dimension $klm(n - 2)$; its homological dimension is only $\min\{k, l, m\}(n - 2)$. This gap was one of the motivations for looking for better descriptions of these path spaces.

In this specific case, we could show that at least the homology and the cohomology of the relevant path spaces $\vec{P}(\tilde{X})(\mathbf{0}, \mathbf{k})$ are, loosely speaking, algebraically generated by the cubical holes in the complex. Every such hole is characterized by the smallest integral vertex $\mathbf{l} \in \mathbf{Z}^n$ above it; this integral vector satisfies $\mathbf{0} \ll \mathbf{l} \leq \mathbf{k}$ (with $\mathbf{a} \ll \mathbf{b} \Leftrightarrow a_i < b_i$ for all i).

Let $Z^*(\tilde{X})(\mathbf{0}, \mathbf{k})$ denote the free graded exterior \mathbf{Z} -algebra with generators the holes $\mathbf{0} \ll \mathbf{l} \leq \mathbf{k}$; every generator has grade $n - 2$. Let $I(\tilde{X})(\mathbf{0}, \mathbf{k})$ denote the ideal generated by products $\mathbf{l}_1 \mathbf{l}_2$ with $\mathbf{l}_1 \not\leq \mathbf{l}_2$ and $\mathbf{l}_2 \not\leq \mathbf{l}_1$. Let $F^*(\tilde{X})(\mathbf{0}, \mathbf{k})$ denote the quotient algebra $F^* = Z^*/I^*$; a (graded) free abelian group with a basis consisting of ordered *cube sequences* $[\mathbf{a}^*] = [\mathbf{0} \ll \mathbf{a}^1 \ll \mathbf{a}^2 \ll \dots \ll \mathbf{a}^r \leq \mathbf{k}]$ (in dimensions $r(n - 2)$) in \tilde{X} .

THEOREM 7.4.1. *Let $n > 2$ and $\mathbf{k} \geq \mathbf{0}$.*

- (1) *Homology $H_*(\vec{P}(\tilde{X})(\mathbf{0}, \mathbf{k}))$ is isomorphic to $F^*(\tilde{X})(\mathbf{0}, \mathbf{k})$ as a graded abelian group.*
- (2) *Cohomology $H^*(\vec{P}(\tilde{X})(\mathbf{0}, \mathbf{k}))$ is isomorphic to $F^*(\tilde{X})(\mathbf{0}, \mathbf{k})$ as a graded ring.*

The proof of Theorem 7.4.1 – first for homology, then refined to cohomology – relies on the fact that the path spaces $\vec{P}(X)(\mathbf{0}, \mathbf{k})$ can be shown to be homotopy equivalent to the *homotopy colimit* of the system of spaces $\vec{P}(X)(\mathbf{0}, \mathbf{k} - \mathbf{j})$ over the poset category \mathcal{J}_n with objects the non-identical binary vectors $\mathbf{j} \in \{0, 1\}^n$, $\mathbf{0} < \mathbf{j} < \mathbf{1}$. Similarly, the graded abelian group $F^*(\tilde{X})(\mathbf{0}, \mathbf{k})$ is a colimit of graded abelian groups $F^*(\tilde{X})(\mathbf{0}, \mathbf{k} - \mathbf{j})$ over the same category. This allows to construct a homomorphism $\Phi : F^*(\tilde{X})(\mathbf{0}, \mathbf{k}) \rightarrow H_*(\vec{P}(\tilde{X})(\mathbf{0}, \mathbf{k}))$.

The final steps of the proof in Raussen and Ziemiański in [RZ14, Section 3] establishing that Φ is indeed a graded isomorphism apply induction using a Bousfield-Kan [BK72] spectral sequence argument. Using Theorem 7.4.1, it is not difficult to calculate the Betti numbers of the relevant path spaces:

COROLLARY 7.4.2. *For $n > 2$ and $\mathbf{k} = (k_1, k_2, \dots, k_n) \geq \mathbf{0}$, Betti numbers are given as*

$$\dim H_{r(n-2)}(\vec{P}(\tilde{X})(\mathbf{0}, \mathbf{k})) = \binom{k_1}{r} \binom{k_2}{r} \dots \binom{k_n}{r}.$$

Homology is trivial in all other dimensions.

For $n = 2$, $\vec{P}(\tilde{X})(\mathbf{0}, (k_1, k_2))$ consists of $\binom{k_1+k_2}{k_1}$ contractible components.

The calculation of the (co-)homology of trace spaces in Theorem 7.4.1 can easily be generalized to a cubical subcomplex of \mathbf{R}^n containing the $(n - 1)$ -skeleton (possibly with fewer holes), cf Raussen and Ziemiański [RZ14]. Further generalizations and other applications of the homotopy colimit constructions are still under consideration.

7.5. Outlook and discussion

Several lines of research are currently under consideration:

Section 7.1: We would like to construct a smaller poset category as a replacement for $\mathcal{C}(X)(\mathbf{0}, \mathbf{1})$ such that the classifying space (nerve) has the same homotopy type. The background is that, in many cases, spaces $X_M, X_{M'}$ may be equal for different matrices $M, M' \in M_{l,n}$.

Section 7.2: So far, we have interpreted the forbidden region F_a corresponding to a semaphore of (general) arity a as a union of hyperrectangles. Instead, one may view the forbidden region $I^n \setminus F_a$ as homotopy equivalent (preserving d-paths and dihomotopies) to the a -skeleton of I^n .

Recent discussions with Fajstrup, Ottosen and Ziemiański show that the path space $\vec{P}(X)(I^n \setminus F_a)(\mathbf{0}, \mathbf{1})$ is homotopy equivalent to a *configuration space*, the so-called “ a -equal manifold” with a topology (in particular its homology) has been studied by Björner and Welker [BW95].

Section 7.4: K. Ziemiański observed recently that it is possible to realize *every!* finite simplicial complex as the trace space of a suitable semaphore model up to homotopy. This result – quite easy to obtain – shows:

- There are no limits to the (combinatorial) “expressiveness” of linear semaphore models.
- It is undecidable whether two HDAs have the same expressiveness.

Related Work. Outlook

8.1. Related Work

Directed Algebraic Topology has been an active research field pursued by a small community for about twenty years. As mentioned in the introduction, the original motivation was an attempt to model and study problems in concurrency theory in Computer Science. The oldest source studying connections between topology and order notions seems to be L. Nachbin’s *Topology and Order* [Nac65] that served as background for our first attempt regarding lpo-spaces. V. Pratt’s idea [Pra91] to use geometric and homotopy notions, later refined by van Glabbeek [vG91], inspired Goubault and Jensen [GJ92] to make use of homology tools. Gunawardena [Gun94] had the first clear-cut application of homotopy methods: a proof that the 2-phase locking method in database engineering is safe; cf Section 2.1.3.

From there on, developments diversified quickly. The following short guide to the literature mentions very briefly the work of several authors on mathematical perspectives motivated by concurrency and not covered in the previous sections. It is certainly biased and non-comprehensive. We start with work published by coauthors:

8.1.1. Approaches by other authors.

Lisbeth Fajstrup: Lisbeth Fajstrup (partially in collaboration with Sokołowski) investigated deadlocks and associated unsafe regions for semaphore models *with* directed loops [Faj00, FS00] and showed that several “de-loopings” may be necessary before the unsafe region is detected correctly. She developed and investigated the directed version of a covering space in [Faj03, Faj10] and gave first results for directed cubical approximation in [Faj05]. The paper [FR08] investigates directed coverings from a categorical perspective. In [Faj14], the author has a close look at the trace space of a torus with holes and relates it to automata theoretic methods.

Éric Goubault, Emmanuel Haucourt, Sam. Mimram and Sanj. Krishnan: Modeling concurrency via Higher Dimensional Automata is originally an idea of Vaughan Pratt’s [Pra91]. It was translated into the mathematical language of (labelled) cubical complexes by Éric Goubault ([GJ92, Gou93, Gou95, Gou01]. We have already taken account of Goubault’s survey article [Gou00] from 2000 followed up by another survey [Gou03] in 2003. Several joint papers with Emmanuel Haucourt [GH05, GH07] and with Sanjeevi Krishnan [GHK09, GHK10] and also by Haucourt [Hau06] are devoted to definitions and properties of *components* in varying contexts and also of categories describing relations them. Krishnan invented

streams in [Kri09] (cf Section 3.1.3) and proved directed simplicial and cubical approximation theorems in [Kri13] (cf Section 3.2.4). The implementation of many concepts and ideas in the software tool ALCOOL is due to Éric Goubault, Emmanuel Haucourt and Samuel Mimram.

Krzysztof Ziemiański: Krzysztof Ziemiański defines (directed) *d-simplicial* complexes and constructs and investigates a pre-cubical model for spaces of *d*-paths or traces in such a complex [Zie12b] – in the same spirit, but more intricate than our construction from Section 7.1. In [Zie12a], he shows that suitable categories of “good” *d*-spaces and of streams are equivalent and enjoy important properties (complete, cocomplete, Cartesian closed). A similar investigation can be found in the article [HHH13] authored by the Hirschowitz family.

Peter Bubenik: Peter Bubenik and Krzysztof Worytkiewicz attempt in [BW06] to reconcile the category of *lpo*-spaces (cf Section 3.1.2) with the tools and techniques of (topological) model categories. In [Bub09], Bubenik studies full subcategories of the fundamental category of a *d*-space with respect to sets of extremal points. In [Bub12], he produces alternative ways of giving trace spaces a combinatorial structure.

Philippe Gaucher: Philippe Gaucher has authored a long series of papers [Gau00, Gau01, Gau02, Gau03d, GG03, Gau03a, Gau03c, Gau03b, Gau05c, Gau05a, Gau05b, Gau06a, Gau06b, Gau07, Gau08a, Gau08b, Gau09] [Gau10a, Gau10b, Gau11] investigating Higher Dimensional Automata and categories of such, mainly from a model category perspective. In particular, he defines categories of *flows* and two types of directed homotopy equivalences, the so-called *S*- and *T*- homotopy equivalences between flows. In recent papers, labelling and higher dimensional transition systems have been taken into consideration, as well.

Marco Grandis: The many research contributions by Marco Grandis have been focused on category theory through many years; most of them are motivated by topological and geometrical considerations in a wide range of set-ups. For several years, he has worked in directed algebraic topology; he is the author of the only book [Gra09] on the subject – written very systematically and with great care. This book builds on many previous articles [Gra03a, Gra02, Gra03b, Gra03c, Gra04, Gra05, Gra06, Gra06b, Gra06a, Gra06c, Gra06, Gra07] without exhausting them.

John F. Jardine: Rick Jardine is a prolific homotopy theorist, amongst others well-known for his book [GJ99] (with P. Goerss) on simplicial homotopy theory. He is mentioned here for his pre-print [Jar02] on *cubical* homotopy theory and for his investigation [Jar10] on path categories leading to algorithmic determinations of fundamental categories.

Thomas Kahl: Thomas Kahl is noted for two contributions [Kah06, Kah12] to directed algebraic topology. In the first, he investigates certain fibration and cofibration category structures on the category of *po*-spaces *under* a given pospace. In the second, he investigates collapsing operations that may reduce the size of a *d*-space without affecting the homotopy type of associated trace spaces.

8.1.2. Distributed Computing and Combinatorial Algebraic Topology.

Distributed Computing is a different area of theoretical Computer Science that

has profited from relations with combinatorial algebraic topology. Roughly speaking, a distributed system is a software system in which components located on networked computers communicate and coordinate their actions by passing messages. A theoretical analysis becomes particularly challenging and interesting when coordination is low; in particular, when participating processors have only private memory and when they can crash without the others being able to observe that.

A typical problem that could be analyzed with tools from combinatorial algebraic topology is the algorithmic unsolvability of the so-called consensus problem (where processors have to agree on one of their inputs). Important contributions to this area earned their authors Herlihy and Shavit [HS99], resp. Saks and Zaharoglou [SZ00] the Gödel prize in 2004, awarded jointly by the European Association for Theoretical Computer Science and the Association for Computing Machinery.

A first source of inspiration for us was the survey book by Herlihy and Rajsbaum [HR95] followed by a long list of articles [HRT98, CHT99, HR99, HS99, HR00, HRT00, CHLT00, GHR06, GHP09, Her10] reasoning with tools from combinatorial algebraic topology on the (non)-solvability of (simplicial) tasks using simplicial protocol complexes extending a given simplicial input complex.

We are looking forward to the comprehensive textbook by Herlihy, Kozlov and Rajsbaum [HKR14] on the subject. It is a challenge to compare and to attempt finding formal relations between the use of tools from combinatorial algebraic topology in Distributed Computing and in Concurrency Theory.

8.1.3. Morse Theory. Relativity Theory. Another sort of inspiration came from the application of ideas from Morse theory applied to relativity theory – with causal or time-like curves playing the role of d-paths. This theme had been covered quite extensively in Penrose’s classical monograph [Pen72] that includes important notions like the domain of dependence.

Rafal Wisniewski’s Ph.d.-dissertation [Wis05] is not in relativity theory, but it applies Morse theory ideas to the analysis of (almost) flow lines of gradient- and non-gradient vector fields. He allows limited perturbations from the flow (within a cone) in the definition of a d-path. A follow up paper by Raussen and Wisniewski appeared as [WR07]. The borderline between dynamical systems and directed topology deserves certainly more attention.

8.2. Outlook: Applied and Computational Algebraic Topology

The research field directed algebraic topology (with an eye to several fields of applications) is one of the areas composing the European research network ACAT: Applied and Computational Algebraic Topology. This network allows to facilitate many of the conferences in the area and to sponsor visits between researchers in the area. The network cooperates with researchers all over the world.

In the United States, applications of algebraic topology within engineering and science disciplines cover already a broad and established research spectrum. This is for instance witnessed by the entire academic year 2013 – 2014 devoted to Scientific and Engineering Applications of Algebraic Topology at the Institute of Mathematics and its Applications in Minneapolis, MN, USA.

Important research activities (apart from directed topology) in the ACAT network include

- Computational Algebraic Topology – including in particular research on persistent homology and its many applications; moreover topological aspects of visualization and shape analysis.
- Topological Robotics
- Stochastic Topology
- Applied Combinatorial Algebraic Topology.

It is impossible to cover these research areas within a few sentences; a list of survey books and articles must suffice here:

- Zomorodian [**Zom05**] and Edelsbrunner and Harer [**EH10**] on computational algebraic topology and, in particular, persistent topology
- Farber [**Far08**] on topological robotics
- Costa, Farber and Kappeler [**CFK12**] on stochastic topology
- Kozlov [**Koz08**] on combinatorial algebraic topology.

Cooperation and cross-fertilization between various subcommunities in applied and computational algebraic topology seem to be of utmost importance for future developments.

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Dansk Resumé

Denne afhandling omhandler bidrag til et nyt forskningsfelt: *Algebraisk Topologi med retning*. Den giver et overblik over sytten forskningsartikler som blev publiceret inden for området i årene 1998 – 2013.

Algebraisk topologi med retning er et forholdsvis nyt forskningsområde. I “klassisk” algebraisk topologi (med sine mange forbindelser til geometri, analyse, algebra, fysik mv.) ræsonnerer man abstrakt i termer af kombinatorik, algebra og kategorier om emner af en geometrisk natur. Når man tager retninger med i betragtning, tillader man *ikke alle* geometriske stier; specielt kan man som regel ikke invertere en sti (“løbe baglæns”). Dette ødelægger muligheden for at oversætte umiddelbart til de fleste vante algebraiske strukturer.

Indtil videre kommer den vigtigste motivation for rum og stier med retning fra teorien om parallelitet (“concurrency”) i den teoretiske datalogi: Når flere processer kan arbejde sig igennem hver sit program uden at der på forhånd er fastlagt en rækkefølge ligger det ikke fjernt at finde på geometriske modeller hvor stierne har en retning: Tiden kan ikke gå baglæns! Et vigtigt spørgsmål som blev behandlet i starten af arbejdet var hvordan man algoritmisk hurtigt finder “deadlocks” og tilhørende usikre regioner hvorfra man ikke kan nå i mål.

Man kan både undersøge specifikke modeller og også mere generelle rum hvis egenskaber det gælder om at kaste lys på. Det giver sig selv at man ofte *ikke umiddelbart* kan bruge metoderne fra den “klassiske” algebraiske topologi til ret meget. Man kan ikke engang definere en fundamentalgruppe for stier med retning. Men nogle af fagets metoder kan alligevel bringes i anvendelse – når først man har stillet spørgsmålene rigtigt.

En stor del af undersøgelserne i afhandlingen handler om at beskrive og analysere *rum* af *d*-stier – stier har en retning, rum af stier har ikke – med udgangspunkt i viden om det såkaldte tilstandsrum som stierne bevæger sig i. Desuden ønskes information om stirummenes indbyrdes sammenhæng givet information om endepunkterne.

Man kan sige en del “kategorielt” om dette emne (se især publikationen [Rau07] for ret generelle *d*-rum). Men når man skal foretage egentlige udregninger af topologiske invarianter for stirum, så lykkes det indtil videre kun for konkrete tilstandsrum; dem som er modeller for parallelle beregninger. Her er det til gengæld lykkedes at beskrive en direkte vej fra en model af tilstandsrummet til beskrivelse af stirummet (udførelser af et parallelt program) som et kombinatorisk beskrevet *simplicial kompleks* (se [Rau10, Rau12a, Rau12b, FGH⁺12]) med relevante topologiske invarianter. Den sidstnævnte artikel beskriver implementeringen af metoden i en praktisk anvendelig algoritme!